16.31 Fall 2005
Lecture Wed 28-Sep-05 ver 1.1

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TODAY:

Finish investigating the structure of $\dot{x} = Ax$ given a transformation $x = Mq$.

GOAL: Diagonalize the system matrix A

- Modal of form of

  $$\dot{x} = Ax + Bu$$
  $$y = Cx + Du$$

  which is

  $$\dot{q} = M^{-1}AMq + M^{-1}Bu \quad (1)$$
  $$y = CMq + Du \quad (2)$$

  is useful for analysis

- We want to find a state variable transformation $x = Mq$ such that A is transformed into diagonal form

  $$\Lambda = M^{-1}AM = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

- A transformation M that will do this is to let the columns of M be the set of eigenvectors x_i of the matrix A.

- We need to show that in the case that the system matrix A has n distinct eigenvalues λ_i, that A is diagonalized into Λ, but more importantly, we must show that the set of eigenvectors x_i are linearly independent, and thus form a basis for the state space.
- We will then show that in the case that A does not have n distinct eigenvalues, that it *may not* be possible to diagonalize A into Λ. In that case A can be placed into Jordan form J, and a set of n linearly independent generalized eigenvectors x_i can be found to form the columns of M.

Review of Basis

- A basis of a vector space V of dimension n is a set of n linearly independent vectors. Any vector in the vector space is then a unique linear combination of the basis vectors. (DeRusso et a., p 75)
- Proof of uniqueness is by contradiction:
  Assume that the representation of a vector $y \in V$ is not unique: That there are two sets of constants $k_1, \ldots, k_n$ and $k_1', \ldots, k_n'$ not all equal to each other such that y is represented by both sets of constants.

$$\begin{aligned} y &= k_1 x_1 + \cdots + k_n x_n && (3) \\ y &= k_1' x_1 + \cdots + k_n' x_n && (4) \\ & \quad \text{Subtract both representations from each other} && (5) \\ 0 &= y - y && (6) \\ &= (k_1' - k_1) x_1 + \cdots + (k_n' - k_n) x_n && (7) \\ x_i &\neq 0 \quad \text{implies} && (8) \\ k_i &= k_i' \quad \text{which is a contradiction} && (9) \end{aligned}$$

- Also note, x_i's linearly independent implies the matrix

$$M = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$$

has full rank $r = n$ and nullity $q = 0$ so that the only vector in it's nullspace is 0.

- Alternative proof:
  Let $k = [k_1, \ldots, k_n]^T$ and $k' = [k'_1, \ldots, k'_n]^T$. Then

$$\begin{aligned} y &= Mk && (10) \\ y &= Mk' && (11) \\ y - y &= M(k - k') && (12) \\ 0 &= M(k - k') && (13) \\ & \text{but the only vector in } N(M) \text{ is 0, so} && (14) \\ (k - k') &= 0 \qquad \text{which is a contradiction} && (15) \end{aligned}$$

Eigenvectors of A form a basis for the State Space

- We need to show that the n distinct eigenvectors x_i each corresponding to the distinct eigenvalues λ_i form a basis, that is, they are linearly independent. Proof by contradiction: Assume that they are not linearly independent

- Proof by contradiction:

$$0 = k_1 x_1 + \cdots + k_n x_n \qquad k_i \text{ not all zero}$$

  Assume $k_1 \neq 0$ and form the matrix

$$(A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I)$$

  . Premultiply both sides of the equation.

$$(A-\lambda_2 I)(A-\lambda_3 I) \cdots (A-\lambda_n I)0 = (A-\lambda_2 I)(A-\lambda_3 I) \cdots (A-\lambda_n I)(k_1 x_1 + \cdots + k_n x_n)$$

  LHS is zero.

$$0 = (A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I)(k_1 x_1 + \cdots + k_n x_n)$$

  RHS

$$\begin{aligned} 0 = \; & k_1 (A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I) x_1 \quad + \cdots + && (16) \\ & k_i (A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I) x_i \quad + \cdots + && (17) \\ & k_n (A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I) x_n && (18) \end{aligned}$$

Distributing x_i

$$0 = k_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n)x_1 \quad + \cdots + \tag{19}$$
$$k_i(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_i)\cdots(\lambda_i - \lambda_n)x_i \quad + \cdots + \tag{20}$$
$$k_n(\lambda_n - \lambda_2)(\lambda_n - \lambda_3)\cdots(\lambda_n - \lambda_n)x_n \tag{21}$$

Leaving

$$0 = k_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n)x_1$$

But $\lambda_i \neq \lambda_j$ for $i \neq j$ implies $k_1 = 0$ which is a contradiction. Q.E.D.

- Same trick applies in the more general case of the Jordan form where the repeated eigenvalues *may* result in a degeneracy of eigenvectors. In that case we create a combination of n linearly independent eigenvalues and generalized eigenvectors, and the proof by contradiction proceeds in a similar manner!

A is Diagonalizable if it has n distinct eigenvalues

- Proof by construction:
  Let M be matrix whose columns are the n linearly independent eigenvectors x_i.
  We want so that
  $$\Lambda = M^{-1}AM$$
  It is easier to equivalently show that
  $$M\Lambda = AM$$
  $$\begin{pmatrix} | & & | \\ x_1 & \dots & x_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$$
  $$\begin{pmatrix} | & & | \\ \lambda_1 x_1 & \dots & \lambda_n x_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ax_1 & \cdots & Ax_n \\ | & & | \end{pmatrix}$$
  $$\begin{pmatrix} | & & | \\ \lambda_1 x_1 & \dots & \lambda_n x_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 x_1 & \dots & \lambda_n x_n \\ | & & | \end{pmatrix}$$

Q.E.D.

Diagonalizability

- Diagonalizability is concerned with the eigenvectors
- (Invertibility is concerned with the eigenvalues!)
- The only connection between eigenvalues and diagonalizability is that diagonalization *can* fail if there are repeated eigenvalues.
- The test is to check, for an eigenvalue that is repeated q times, whether there are q independent eigenvectors.
- Examples

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \qquad \text{Repeated positive e-values. Not diagonalizable} \tag{22}$$

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \text{Repeated positive e-values. Diagonalizable!} \tag{23}$$

$$0 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad \text{Repeated zero e-values. Diagonalizable! Not invertible!} \tag{24}$$

Jordan Form

- If a matrix A has s linearly independent eigenvectors, then it is similar to a matrix in Jordan form with s square blocks on the diagonal:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{pmatrix}$$

- Each block J_i has one eigenvector, one eigenvalue, and 1's just above the diagonal.

$$J_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix}$$

- J is equal to
$$J = M^{-1}AM$$
where the columns M contain the s linearly independent eigenvectors and $n-s$ generalized eigenvectors.

- This implies
$$AM = MJ$$
or
$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ x_1 & \ldots & x_n \\ | & & | \end{pmatrix} \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{pmatrix}$$

- Multiplying through we find that
$$Ax_i = \lambda_i x_i \qquad \text{or} \qquad Ax_i = \lambda_i x_i + x_{i-1}$$

- DeRusso et al., pp 140–143, gives an algorithm for calculating generalized eigenvectors.

- Each J_i can also be written as $\Lambda_i + N_i$ where N_i is a matrix of zeros along the diagonal and 1's along the superdiagonal.
$$J_i = \Lambda_i + N_i$$
$$\begin{pmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{pmatrix} = \begin{pmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{pmatrix}$$

- For argument lets say that J_i, Λ_i, and N_i are $n \times n$ matrices and drop the i subscript for now. Then
$$N^n = 0$$

- We can now calculate e^{Jt}
$$\begin{aligned} e^{Jt} &= e^{(\Lambda+N)t} && (25) \\ &= e^{\Lambda t} e^{tN} && (26) \\ &= e^{\lambda t} \sum_{k=0}^{n-1} \frac{t^k N^k}{k!} && (27) \end{aligned}$$

or in matrix form

$$e^{Jt} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & 1 & t & & t^{n-2}/(n-2)! \\ & & 1 & & \\ & & & \ddots & \\ & & & & t \\ & & & & 1 \end{pmatrix}$$

- We will rarely use Jordan form in class, but it is important to
  1. Know that it exits and that it is the most general form of diagonalization
  2. Know and understand the form of e^{Jt}
- *However*, the Jordan form does lead us to an interesting discussion about system models and numerical computation with system models

FRIDAY:

- Look at an interesting modeling/numerical issue that can arise and is related to our discussion on diagnonalization and Jordan forms.
- Summarize transformations
- Find a better way to calculate e^{At} given that it shows up so much in the solutions to

$$\dot{x} = Ax + Bu \quad (28)$$

$$y = Cx + Du \quad (29)$$