16.31 Fall 2005
Lecture Wed 5-Oct-05

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TODAY

- We are working on the following learning outcomes today:
  - Determine whether a realization is minimal
  - Perform a Kalman decomposition and reason about it!
  - Analyze system property effects on controllability
- Approximate reading coverage:
  - DeRusso, Section 6.8
  - Bélanger, Sections 3.7.4, 3.7.5, 3.8

State Space Realizations (A,B,C)

State Space Realizations:

- A state space model A, B, C corresponding to a transfer function G is known as a realization of the transfer function.

Fun Facts:

- Every state space model has a unique transfer function.
- Every transfer function has an infinite number of possible state space realizations

Reasons for non-uniqueness

Fun Facts:

- Every state space model has a unique transfer function.
- Every transfer function has an infinite number of possible state space realizations

Reasons for non-uniqueness of state space realization:

- State variables may be defined which do not affect the transfer function. (Kalman Decomposition)
- The transfer function is invariant under non-singular linear transformations ($x = Mq$).

Transformation Invariance

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$G(s) = C(sI - A)^{-1}B + Du$$

$$x = Mq$$

$$\begin{aligned} \dot{q} &= M^{-1}AMq + M^{-1}Bu \\ y &= CMq + Du \end{aligned}$$

$$G(s) = ???$$

Transformation Invariance

$$\begin{aligned} \dot{q} &= M^{-1}AMq + M^{-1}Bu \\ y &= CMq + Du \end{aligned}$$

$$G(s) = ???$$

$$\begin{aligned} G(s) &= CM(sI - M^{-1}AM)^{-1}M^{-1}Bu + Du \\ &= CM(sM^{-1}IM - M^{-1}AM)^{-1}M^{-1}Bu + Du \\ &= CM[M^{-1}(sI - A)M]^{-1}M^{-1}Bu + Du \\ &= CM[M^{-1}(sI - A)^{-1}M]M^{-1}Bu + Du \\ G(s) &= C(sI - A)^{-1}Bu + Du \end{aligned}$$

Variables which do not affect the transfer function

- Any realization A_1, B_1, C_1 based on a set of variables x_1 can be augmented by state variables that do not affect the transfer function:

$$G(s) = C_1(sI - A_1)^{-1}B_1$$

- Unobservable states x_2 can be added which have no effect on the original states x_1 and no direct effect on the output y.

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 x_1 \end{aligned}$$

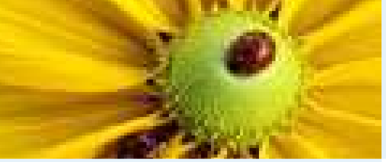
Variables which do not affect the transfer function

- Minimal system transfer function

$$G(s) = C_1(sI - A_1)^{-1}B_1$$

- Uncontrollable states x_3 can be added which are not affected by the original states x_1 nor by the input u.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_1 & A_{13} \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$



Variables which do not affect the transfer function

- Minimal system transfer function

$$G(s) = C_1(sI - A_1)^{-1}B_1$$

- Here is a realization that includes the addition of both unobservable states x_2, and uncontrollable states x_3

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & A_{13} \\ A_{21} & A_2 & A_{23} \\ 0 & 0 & A_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} u$$

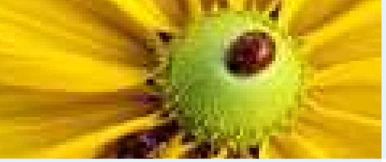
$$y = \begin{pmatrix} C_1 & 0 & C_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Variables which do not affect the transfer function

- Minimal system transfer function

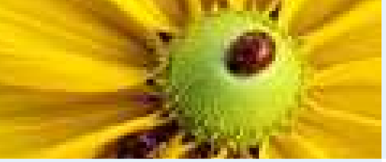
$$G(s) = C_1(sI - A_1)^{-1}B_1$$

- We could have added a set of variables x_4 that were both unobservable and uncontrollable. You might want to try this at home!



Kalman's Decomposition

- Following the above argument, you might be able to say that in general, a realization can be paroned into four subsets:
  1. States which are controllable and observable
  2. States which are controllable but unobservable
  3. States which are uncontrollable but observable
  4. States which are both uncontrollable and unobservable
- This result is due to Kalman and we will prove its existence in a later lecture.



Minimal Realizations

- Given that we can add, or a system may possess, additional states that are uncontrollable, unobservable, or both, we might be interested in defining a minimal realization.
- Definition: A realization is minimal if the number of states is the same as the order of the transfer function.

More Fun Facts:

- The number of states which are both controllable and observable is the same as the order of the transfer function.
- A realization that is minimal is always both controllable and observable.

Controllability and Observability - Review

- Complete Controllability:
  The system

  $$\dot{x} = Ax + Bu$$

  is said to be completely controllable if for $x(0) = 0$ and any given state x_1 there exists finite time t_1 and a piecewise continuous input $u(t)$, $0 \le t \le t_1$ such that $x(t_1) = x_1$.
- Complete Observability:
  The system

  $$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

  is said to be completely observable if there is a $t_1 > 0$ such that knowledge of $y(t)$, for all t, $0 \le t \le t_1$, is sufficient to determine $x(0)$.

Why is any of this important?

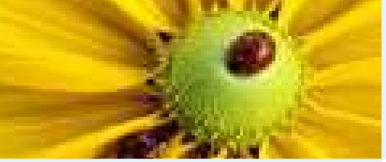
- Controllability will imply that we can design a controller applying state feedback control u=-Kx to modify all of the system responses to our liking.
- Observability will imply that we can design an observer using observer gains L to observe all of the states of the system that we do not measure.

Controllability & Observability: Too much?

- Although complete controllability and observability are desirable, they are not an essential aspect of the DESIGN in all applications.
- We may only require that any uncontrollable or unobservable states be stable - that they tend to zero as time increase.
- This leads to the concepts of stabilizability and detectability which are suitably weaker than complete controllability and complete observability.

Stabilizability and Detectability

- Stabilizability:
  A set of dynamic state equations is said to be stabilizable if any uncontrollable states in the set are stable.
- Detectability:
  A set of dynamic and output equations is said to be detectable if any unobservable states in the set are stable.



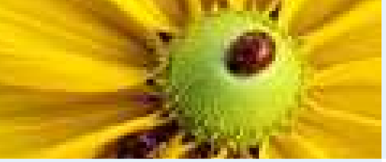
Controllability and Observability - Real Life Designs

- So far we've had a lot of theoretical definitions, but what can cause unobservability or uncontrollability in a mathematical model of a real system?
  1. In general, a system contains real internal variables which are not accessible to either or both control or observation.
  2. The relationship between states and either or both inputs and outputs may not be linearly independent.
  3. A system could have subsystems having identical dynamics.
  4. Pole-zero cancellation in the transfer function of cascaded systems.
- We should be concerned about these issues in the up front DESIGN of a controlled system!

Next

- Trivial Example: The car!
- FRIDAY:

  1. Less trivial example: Satellite control
  2. (maybe) More examples of loss of controllability/observability
  3. More proofs! :)



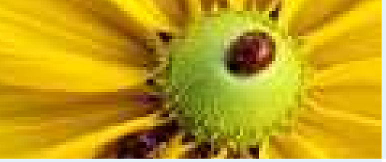
Toy Problem

- Consider the following system:

$$
\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + u \\
\dot{x}_2 &= \lambda_2 x_2 + u \\
\dot{x}_3 &= \lambda_3 x_3 \\
\dot{x}_4 &= \lambda_4 x_4 \\
y &= x_1 + x_3
\end{aligned}
$$

$$
A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$

$$
C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 \end{pmatrix}
$$



Toy Problem Transfer Function

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B \\
&= \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{s-\lambda_1} & 0 & 0 & 0 \\ 0 & \frac{1}{s-\lambda_2} & 0 & 0 \\ 0 & 0 & \frac{1}{s-\lambda_3} & 0 \\ 0 & 0 & 0 & \frac{1}{s-\lambda_4} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
G(s) &= \frac{1}{s-\lambda_1}
\end{aligned}
$$

- There are four (4) states in the realization of the transfer function $G(s)$, but the order of the transfer function is one (1), so the toy problem realization is not a minimal realization.

Toy Problem Minimal Realization

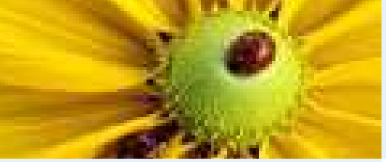
- A minimal realization for the transfer function G(s) would have one (1) state. And this state would be both controllable and observable.
- A candidate minimal state realization for the transfer function is

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + u \qquad & A &= (\lambda_1) \quad B = (1) \\ y &= x_1 & C &= (1) \quad D = (0) \end{aligned}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s - \lambda_1}$$

- $G(s)$ is of order 1 and the state space realization has one state, thus this realization is a minimal realization. Its one state is both controllable and observable.



Improving the Toy Problem with State Feedback

- Perhaps we don't like the eigenvalues of the system matrix. Can we "improve" the response of the system with full state feedback?

$$u = -Kx = -\begin{pmatrix} k_1 & k_2 & k_3 & k_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- With full state feedback the toy system becomes

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 - (k_1 x_2 + k_2 x_2 + k_3 x_3 + k_4 x_4) \\ \dot{x}_2 &= \lambda_2 x_2 - (k_1 x_2 + k_2 x_2 + k_3 x_3 + k_4 x_4) \\ \dot{x}_3 &= \lambda_3 x_3 \\ \dot{x}_4 &= \lambda_4 x_4 \\ y &= x_1 + x_3 \end{aligned}$$

Improving the Toy Problem Response

$$\dot{x} = (A - BK)x$$

$$\begin{aligned}
(A - BK) &= \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} - \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 - k_1 & -k_2 & -k_3 & -k_4 \\ -k_1 & \lambda_2 - k_2 & -k_3 & -k_4 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}
\end{aligned}$$

Toy Problem Full State Feedback

$$
\begin{aligned}
&\det[sI-(A-BK)] \\
&= \begin{vmatrix} s-(\lambda_1-k_1) & -k_2 & -k_3 & -k_4 \\ -k_1 & s-(\lambda_2-k_2) & -k_3 & -k_4 \\ 0 & 0 & (s-\lambda_3) & 0 \\ 0 & 0 & 0 & (s-\lambda_4) \end{vmatrix} \\
&= (s-\lambda_4)(s-\lambda_3) \begin{vmatrix} s-(\lambda_1-k_1) & -k_2 \\ -k_1 & s-(\lambda_2-k_2) \end{vmatrix}
\end{aligned}
$$

- We can only change the placement λ_1 and λ_2. Eigenvalues λ_3 and λ_4 are fixed.
- Why feedback states x_3 and x_4 anyway?
- In this case we might only require that the system is stabilizable and detectable.

The car

- States

$$d : \text{distance} \qquad x_1$$
$$v : \text{velocity} \qquad x_2$$

- Dynamics

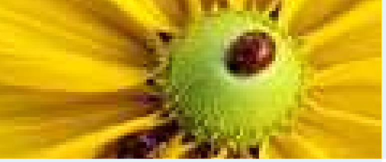
$$F = ma = m\dot{x}_2 \qquad m = 1$$

- Control

$$u = F$$

- Output

$$y = d = x_1$$



Car State Equations

- State Equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

- System Matrix A is singular and is in Jordan form

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1}$$

State Transition Matrix e^{At}

- e^{At}

$$\begin{aligned} e^{At} &= I + At + [0]t^2/2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

- Zero-Input Response

$$\begin{aligned} x(t) &= e^{At} x(0) \\ \begin{pmatrix} d(t) \\ v(t) \end{pmatrix} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ v_0 \end{pmatrix} \\ d(t) &= d_0 + v_0 t \\ v(t) &= v_0 \end{aligned}$$

Design: Odometer, Speedometer, or both?

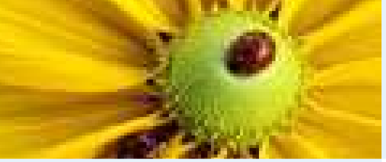
- Sensors cost! Do I need both an odometer and a speedometer?
- Odometer and speedometer imply both states measured

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

- Check observability

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

- rank 2 so we are OK!



Design: Speedometer only?

- Speedometer only implies only x_2 measured

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

- Check observability

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

- rank 1 so only paying for a speedometer would be a bad design decision.

Design: Odometer only?

- Odometer only implies only x_1 measured

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

- Check observability

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}$$

- rank 2! We can get away with only paying for an odometer.
- So why do cars have both an odometer and a speedometer?
- Is there any reason that we would like to "estimate" the states even if we have sensors?