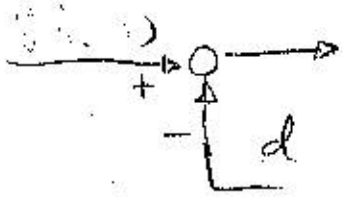
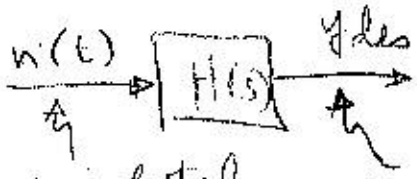
LQ tracking problem (Section 7.5.1)



If y_{des} is time varying, can we optimize the tracking properties?

Need to characterize y_{des}. We do this using a "shaping filter" driven by white noise.



NOTE: if $H(s) = 1$, LQR is solution for this input

completely random

bandwidth and perhaps frequency content, properties (follow a standard)

Approach:

① Create $H(s)$ in state space:

$$\dot{z}_y = A_y z_y + B_y w$$
$$y_d = C_y z_y$$

② Append systems:

$$\begin{bmatrix} \dot{x} \\ \dot{z}_y \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & A_y \end{bmatrix}}_{A_{aug}} \begin{bmatrix} x \\ z_y \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_{aug}} u + \begin{bmatrix} 0 \\ B_y \end{bmatrix} w$$

$$y = \begin{bmatrix} C & -C_y \end{bmatrix} \begin{bmatrix} x \\ z_y \end{bmatrix} = C_{aug} x_{aug}$$

(3) Choose $J = \int_0^\infty y^T Q y + u^T R u$

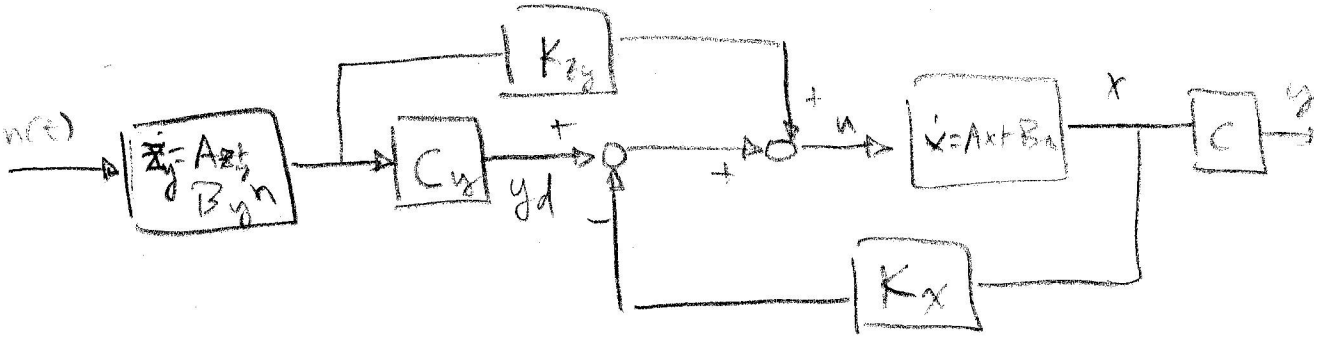
$= \int_0^\infty x_{aug}^T C_{aug}^T Q C_{aug} x_{aug} + u^T R u$

(4) Solve for LQR Gains:

$K = LQR(A_{aug}, B_{aug}, Q, R)$

$= \begin{bmatrix} K_x & K_{z_y} \end{bmatrix}$

NOTE: A_y dynamics don't change



Note: you usually don't have the states of z_y available!

If we build an estimator for z_y, then the choice of A_y and B_y will create a compensator in our feedback system

Need to consider Estimation.

Observers Section 7.6 Belanger

Problem: given $\dot{x} = Ax + Bu$

$y = Cx$

how can you estimate $x(t)$ based on $y(t)$ [and knowledge of A, B, C?]

If we can do this, we can do $u = -K\hat{x}$ where $\hat{x}$ is the estimated state.

Solution: Start w/ the obvious estimate.

$$\dot{\hat{x}} = A\hat{x} + Bu$$

look at $\tilde{x} = (x - \hat{x})$ the error;

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + B(u - u)^{0}$$

$$\dot{\tilde{x}} = A\tilde{x}$$

If A is stable, $\tilde{x} \to 0$ over time.

BUT. what if damping is poor?

what if $\hat{B} \neq B$? NEED FEEDBACK OF y INFO

Luenberger Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + G(y - C\hat{x})$$

If y is bigger than $C\hat{x}$, drive $\hat{x}$ downward ($\dot{\hat{x}} < 0$), etc.

Now: $\dot{\tilde{x}} = A\tilde{x} - G(Cx - C\hat{x})$

$Cx - C\hat{x} = C\tilde{x}$

$$\dot{\tilde{x}} = (A - GC)\tilde{x}$$

Two benefits: ① we can set the response time
② we introduce robustness by feeding back measurements

Note: $\text{Eig}(A - GC) = \text{Eig}(A^T - C^T G^T)$

Setting pole locations is equivalent to designing feedback controller for

$$\dot{x}_D = A^T x_D + C^T u_D$$

G^T is "Full state feedback" for this system.

Dual optimality Problem yields K.F.

Simple Observer Design

① Find A, C

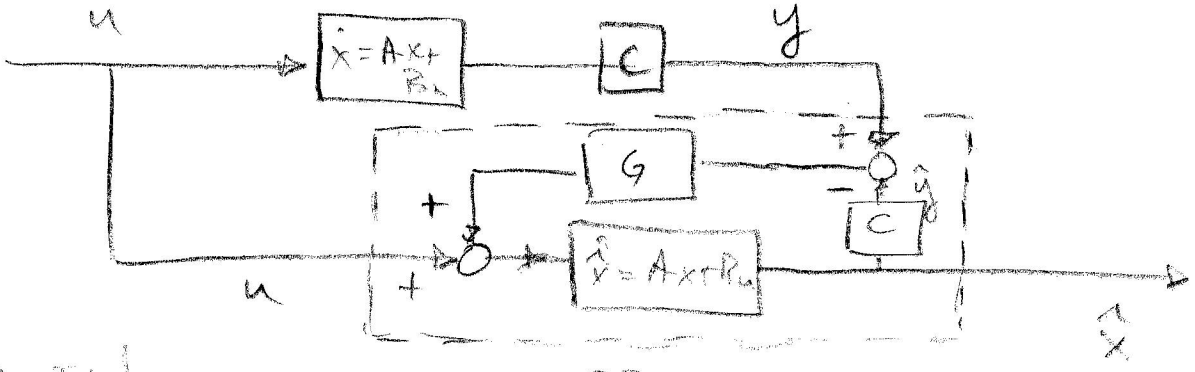
② Perform Pole Placement on A^T, C^T

→ OBSERVER:

③ $\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy$

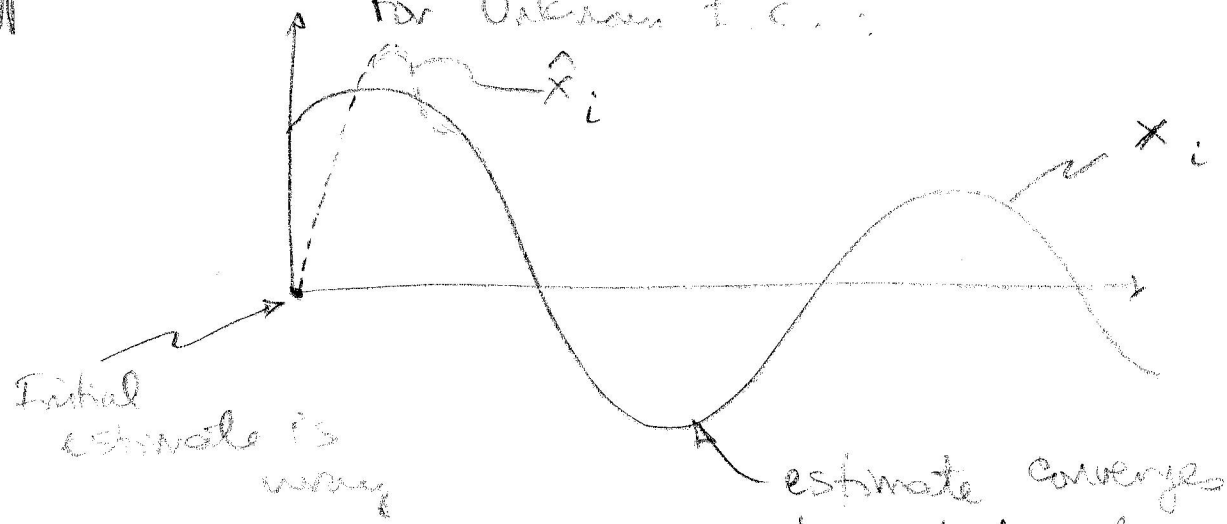
Note: (A^T, C^T) Needs to be Controllable

IOWs (A, C) Observable!



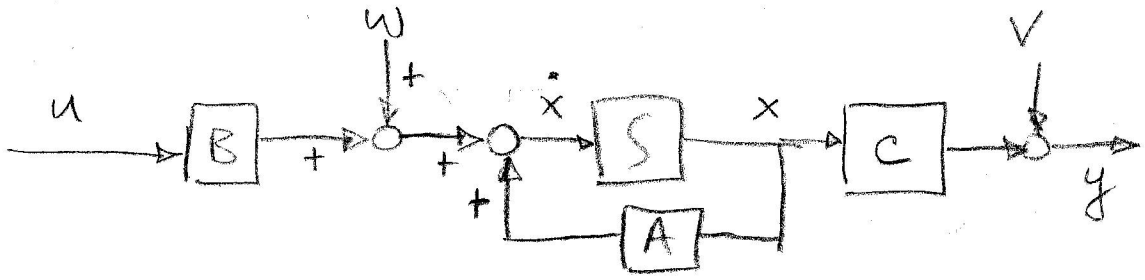
OBSERVER

Typical Behavior For Unknown I.C.:



(if A in observer is exactly right, and sensor is unbiased)

How do we determine an optimal observer? Must consider the effects of Noise. More detailed plant model:



$$\dot{x} = Ax + Bu + w$$
$$y = Cx + v$$

w is called "process noise" it actually affects the state, so we want to track that effect.

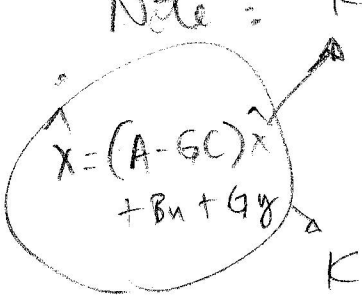
v is called "measurement" or "sensor" noise. It is error on the value of y that we are feeding back — ruins our ability to determine $\hat{x}$ = want to reject

$$\dot{\hat{x}} = A\hat{x} + Bu + G(y - C\hat{x})$$

G small = y is poor, rely on u and $A\hat{x}$ (interacting)

G large = u is not only thing driving $\hat{x}$, rely more on y

Note: K small → Low Bandwidth



on feedback, rely on natural dynamics and measured inputs

K large – High Bandwidth

Noisy measurement ⇒ Noisy estimate

S-S Kalman Filter

given:

$$\dot{x} = Ax + Bu + B_w w$$

$$y = Cx + v$$

(A, C) Detectable

w, v white noise w/ intensity matrices $W \neq V$

Typically, W and V are diagonal,

and $W_{ii} = (RMS)^2$ of process noise (variance)

$V_{ii} = (RMS)^2$ of sensor noise (variance)

goal: minimize variance $\left(\tilde{x}^T \tilde{x}\right) = E\{\Sigma \tilde{x}^2\}$

Soln:

$$G = PC^T V^{-1}$$

where P is the pos def sym. soln of

$$AP + PA^T + B_w W B_w^T - PC^T V^{-1} CP = 0$$

see book for derivation based on duality

Note: w/ the plant shown, we have

for the estimator:

$$\dot{x} - \dot{\hat{x}} = [Ax + Bu + B_w w] - [A\hat{x} + Bu + G(Cx + v - C\hat{x})]$$

$$= (A - GC)\hat{x} + B_w w + Gv$$

If G is larger → tracks $B_w w$ faster

But Gv corrupts more!

K.F. makes this trade optimally

Consider SISO case:

$$\dot{\hat{x}} = (a - gc)\hat{x} + \frac{g}{s} y + bu$$

$$\hat{x} = \frac{\frac{g}{s}}{s - (a - gc)} y + \frac{b}{s - (a - gc)} u$$

- Low-pass filter w/ pole at $(a - gc)$
- g larger → high cut-off freq.
- influence of b gets smaller

$\dot{x} = -ax + bu + w \qquad W = 1$ M.S. of w

$y = x + v \qquad V = V$ M.S. of V

$$-2ap + 1 - \tfrac{1}{V}p^2 = 0$$

$$p^2 + 2avp - V = 0 \qquad p = \frac{-2aV \pm \sqrt{4a^2V^2 + 4V}}{2}$$

$$p = -aV + \sqrt{a^2V^2 + V}$$

$$g = \frac{pc}{V} = -a + \sqrt{a^2 + \tfrac{1}{V}}$$

as $V \to 0$ (process noise dominates $g \to \infty$)

as $V \to \infty$ (sensor noise dominates $g \to 0$)

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega$$

$$y = x_1 + v \qquad w = 1, \ v \text{ varies}$$
$$= [1 \quad 0]\, x + v$$

we are sensing position, x_1, but $\dot{x}_1 = x_2$ is not measured

Can show that: $G \cong \begin{bmatrix} 2/\sqrt{r} \\ 2/r - w_0^2 \end{bmatrix}$

$$\dot{\hat{x}} = (A - GC)\hat{x} + Gy$$

$$\dot{\hat{x}} = \begin{bmatrix} 2/\sqrt{r} & 1 \\ 2/r & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 2/\sqrt{r} \\ 2/r - w_0^2 \end{bmatrix} y$$

show? HANDOUT?

$$\hat{X}(s) \cong \frac{s - \sqrt{r}\, w_0^2}{s^2 + \frac{2}{\sqrt{r}} s + \frac{2}{r}} y(s)$$

poles follow R.S.L. as before

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