GOALS OF THIS LECTURE

to understand: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ $\Rightarrow$ $\boxed{\text{THE CONCEPT OF STATE}}$

to define terminology and describe the bookkeeping for

Linear, Time-Invariant Systems

Consider:

$$M\ddot{x} + Kx = F$$

How many states?
why?

concept of state: if we know $x(0)$ $\dot{x}(0)$ and $F(t)\ \forall t \geq 0$
we know everything about the system
for all future time

x is not enough: if mass is moving,
then $x(t)$ $t>0$ is different

the state, $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, remembers all past time
sufficiently to predict all
future time

$\dddot{x} = f$ $\Rightarrow$ need 3 I.C.'s to sol...
$\Rightarrow$ 3 states

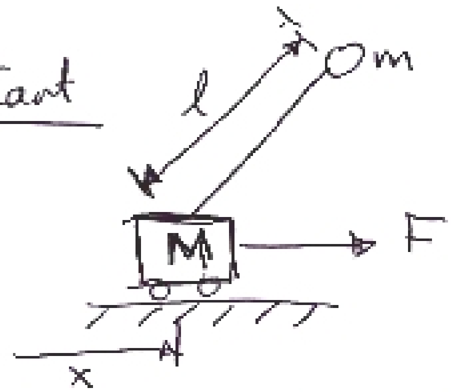
IS This system linear?
IS This system time-invariant?

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad \dot{\underline{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} F$$

Questions? Seen it all?

Inverted Pendulum on a Cart



$$(M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F$$

$$\ddot{x}\cos\theta + l\ddot{\theta} - g\sin(\theta) = 0$$

How many states? Why? linear? T.I?

$$\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} \Rightarrow \text{this is an } \underline{x}$$

For these states, eqns are of the form:

$$f(\dot{\underline{x}}, \underline{x}, \underline{u}) = 0$$

slightly more general than form discussed in the book.

$$\underline{x} = \underline{x}^* + \Delta\underline{x}, \quad \underline{u} = \underline{u}^* + \Delta\underline{u}$$

To linearize: $\dot{\underline{x}} = \cancel{\dot{\underline{x}}^*}^{\,0} + \Delta\dot{\underline{x}}$

$\boxed{\text{PICK an Equilibrium } f(0, \underline{x}^*, \underline{u}^*) = 0}$ substitute into eqns:

$$f(\dot{\underline{x}}, \underline{x}, \underline{u}) = \cancel{f(0, \underline{x}^*, \underline{u}^*)}^{\,0\ \text{(equilibrium)}} +$$

$$\left.\frac{\partial f}{\partial \dot{x}_1}\right|_* \Delta\dot{x}_1 + \left.\frac{\partial f}{\partial \dot{x}_2}\right|_* \Delta\dot{x}_2 + \cdots$$

$$\left.\frac{\partial f}{\partial x_1}\right|_* \Delta x_1 + \left.\frac{\partial f}{\partial x_2}\right|_* \Delta x_2 + \cdots$$

$$\left.\frac{\partial f}{\partial u_1}\right|_* \Delta u_1 + \cdots + \text{H.O.T.'s}$$

we perform the linearization about an equilibrium that we sometimes must find to determine $\frac{\partial f}{\partial x}\Big|_*$ evaluated at the equilibrium

Apply to ~~an~~ ~~ir~~ cart-pendulum:

Equil. $\quad x=0,\ \dot{x}=0,\ \theta=0,\ \dot{\theta}=0$

$\Rightarrow F=0 \quad$ (sometimes much harder)

$$\overbrace{(M+m)}^{\partial f/\partial \dot{x}_3|_*}\ \overbrace{\Delta\ddot{x}}^{\Delta\dot{x}_3} + \overbrace{ml\cos(\theta^*)}^{\partial f/\partial \dot{x}_4|_*}\ \overbrace{\Delta\ddot{\theta}}^{\Delta\dot{x}_4} + \overbrace{ml\ddot{\theta}^*(-\sin\theta^*)}^{\partial f/\partial x_2}\ \overbrace{\Delta\theta}^{\Delta x_2}$$

$$- ml\sin(\theta^*)\,2\dot{\theta}^*\Delta\dot{\theta} - ml(\dot{\theta}^*)^2\cos(\theta^*)\Delta\theta = \Delta F$$

$$\boxed{\begin{aligned}(M+m)\Delta\ddot{x} + ml\Delta\ddot{\theta} &= \Delta F\\ \Delta\ddot{x} + l\Delta\ddot{\theta} - g\Delta\theta &= 0\end{aligned}}$$

$$\begin{bmatrix} M+m & ml \\ 1 & l \end{bmatrix}\begin{bmatrix}\Delta\ddot{x}\\ \Delta\ddot{\theta}\end{bmatrix} = \begin{bmatrix}\Delta F\\ g\Delta\theta\end{bmatrix}$$

$$\begin{bmatrix}\Delta\ddot{x}\\ \Delta\ddot{\theta}\end{bmatrix} = \underbrace{\frac{1}{(M+m)l - ml}}_{\frac{1}{Ml}}\begin{bmatrix} l & -ml \\ -1 & M+m\end{bmatrix}\begin{bmatrix}\Delta F\\ g\Delta\theta\end{bmatrix} = \begin{bmatrix} 1/M & -m/M \\ -1/Ml & \frac{M+m}{Ml}\end{bmatrix}\begin{bmatrix}\Delta F\\ g\Delta\theta\end{bmatrix}$$

give me a linear, TV situation $\Rightarrow M(t) \rightarrow$ AIRCRAFT ROCKETS

goal: $\quad \dot{\underline{x}} = A\underline{x} + B\underline{u}$

How about going up and down hills?

drop Δ's to simplify writing things... ④

$$\underline{x} = \begin{bmatrix} \Delta x \\ \Delta \theta \\ \Delta \dot{x} \\ \Delta \dot{\theta} \end{bmatrix} \qquad \dot{\underline{x}} : \quad \begin{aligned} \Delta \dot{x} &= \underline{x}_3 \\ \Delta \dot{\theta} &= \underline{x}_4 \\ \Delta \ddot{x} &= -\frac{mg}{M} \underline{x}_2 + \frac{1}{M} \underline{u} \\ \Delta \ddot{\theta} &= \frac{(M+m)g}{M\ell} \underline{x}_2 + \frac{1}{M\ell} \underline{u} \end{aligned}$$

$$\underline{u} = \Delta F$$

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{M\ell} & 0 & 0 \end{bmatrix}}_{\text{"A" matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/M \\ 1/M\ell \end{bmatrix}}_{\text{"B" matrix}} \underline{u}$$

IS THIS UNIQUE?

IS THERE MORE OR LESS STATES?

- States may or may not be measured quantities
- (although knowing every state would be great)

— Eventual goal will be to estimate all state and feed back - but measurements will virt. always be different (see JOE paper)

Output equation:

$$\underline{y} = C\underline{x} + D\underline{u}$$

if measurements are $\theta, \dot{\theta}$:

$$\underline{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + [\emptyset]\, \underline{u}$$

$\underline{C}$: any linear combination of states

accelerometer : measures derivative of the state!

$$\underline{y} = \left[0 \quad -\frac{mg}{M} \quad 0 \quad 0 \right] \underline{x} + \left[1/M \right] \underline{u}$$

D matrix is indicative of this type of situation (somewhat non-physical)