Properties of e^{At} "The Matrix Exponential"

First, notation:

Linear, for time-varying systems, transition matrix is the matrix that transitions the state from t_0 to t:

$$x(t) = \phi(t, t_0)\, \underline{x}(t_0)$$

for time-invariant systems, $\phi(t, t_0) = \phi(t - t_0)$

finally since t_0 is irrelevant in TI systems:

set $t_0 = 0$

$$\underline{x}(t) = \phi(t)\, \underline{x}(0)$$

$$\phi(t) \triangleq e^{At} \quad \text{for LTI systems given by } \dot{\underline{x}} = A\underline{x} + Bu$$

$\phi(t - t_0) = e^{A(t - t_0)}$ is fine too . . .

Properties of e^{At}

① $$\left. e^{At} \right|_{t=0} = I, \qquad \frac{d}{dt}\left\{ e^{At} \right\} = A e^{At}$$ } one definition

② $$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \cdots$$

also a possible definition, by analogy to e^{at}

③ if $\dot{x} = Ax$, $x(0) = x_0$

$x(t) = e^{At} x_0$ } could also define e^{At} this way!

also, if $x(t_0) = x_0$

$x(t) = e^{A(t - t_0)} x_0$

(3A) $e^{At} = \mathcal{L}^{-1}\left\{ (Is - A)^{-1} \right\}$

(4) if $\dot{x} = Ax + Bu \quad x(0) = x_0$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

proof: by substitution;
by convolution
by Laplace x form

(5) if $y = Cx$, $\quad y = Ce^{At} x_0 + \int_0^t \underbrace{Ce^{A(t-\tau)} B}_{h(t-\tau)} u(\tau)\, d\tau$

where $h(t) = Ce^{At} B$

~~h(t~~

$h(t)$ is impulse response
(so that output is convolution of input w/ step response)

(6) $e^{\Lambda t}$, where $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ $\quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$

(7) $e^{A(t+s)} = e^{At} e^{As}$ $\quad$ (7A) $e^{(A+B)t} \neq e^{At} e^{Bt}$ unless $AB = BA$

(8) $(e^{At})^{-1} = e^{-At}$

(9) $(e^{At})^n = e^{Ant}$

(10) $e^{A(t_2 - t_1)} e^{A(t_1 - t_0)} = e^{A(t_2 - t_0)}$