Consider a system w/ M inputs, K outputs, and N states (1)

$$\underset{\{N\times1\}}{\dot{x}} = \underset{\{N\times N\}}{A}\underset{\{N\times1\}}{x} + \underset{\{N\times M\}}{B}\underset{\{M\times1\}}{u}$$

$$\underset{\{K\times1\}}{y} = \underset{\{K\times N\}}{C}\underset{\{N\times1\}}{x} + \underset{\{K\times M\}}{D}\underset{\{M\times1\}}{u}$$

$$y(s) = \begin{bmatrix} g_{11}(s) \cdots g_{1M} \\ \vdots \\ g_{K1}(s) \cdots g_{KM} \end{bmatrix} u$$

If the eigenvalues of A are distinct, and v_n are defined as solns of:

$$A v_n = \lambda_n v_n$$

then we can write:

$$A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_N \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_N v_N \end{bmatrix}$$

$$AV = V\Lambda$$

$$V^{-1}AV = \Lambda$$

also let

$$W \triangleq V^{-1} \triangleq \begin{bmatrix} - & w_1^T & - \\ - & w_2^T & - \\ & \vdots & \\ - & w_N^T & - \end{bmatrix}$$ "left eigenvectors"

then for $x = Vz$:

$$\dot{z} = WAVz + WBu$$
$$y = CVz + Du$$

$$\left.\begin{aligned} \dot{z} &= \Lambda z + WB \\ y &= CVz + Du \end{aligned}\right\} \text{ Diagonalized System}$$

Now,

write the effect of the m^{th} input on the k^{th} output (to help understand controllability/observability):

Define: $\left\{ u = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_K \end{bmatrix} \quad B = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_M \\ | & & | \end{bmatrix} \quad C = \begin{bmatrix} - c_1^T - \\ \vdots \\ - c_K^T - \end{bmatrix} \right.$

n^{th} row of system dynamics due to m^{th} input:

$$\circledast \qquad \dot{z}_n = \lambda_n z_n + w_n^T b_m u_m$$

k^{th} output:

$$y_k = \begin{bmatrix} - c_k^T - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_N \\ | & | & & | \end{bmatrix} \underline{z} + d_{km} u_m$$

$$= \sum_{n=1}^{N} c_k^T v_n z_n + d_{km} u_m$$

but from $\circledast$

$$z_n(s) = \frac{1}{s+\lambda_n} w_n^T b_m u_m(s)$$

$$\therefore \quad y_K(s) = \sum_{n=1}^{N} \frac{(c_K^T v_n)(w_n^T b_m)}{(s+\lambda_n)} u_m(s) + d_{km} u_m(s)$$

$$y_K = \sum_{n=1}^{N} \frac{(c_K^T v_n)(w_n^T b_m)}{(s+\lambda_n)} u_m + d_{km} u_m$$

$\{=$ partial fraction expansion of $g_{km}(s)\}$

$(c_K^T v_n)$ is a scalar which indicates the degree of observability of the n^{th} mode (λ_n) when observed from the K_{th} output

$(w_n^T b_m)$ is a scalar indicating the degree of controllability of the n^{th} mode from the m^{th} input

(d_{km}) may provide additional "controllability" or "observability" of the m^{th} input from the K^{th} output, but does not help in controlling or observing the modes λ_n.

NOTE:

if $(c_k^T v_n) = 0 \quad \forall k$

or, I.O.W.'s $C v_n = 0$, then mode n is unobservable

if $(w_n^T b_m) = 0 \quad \forall m$

or, I.O.W.'s $w_n^T B = 0$, then the mode n is uncontrollable

$\Rightarrow$ adding a sensor (row in C matrix) can recover observability (if $c_{new}^T v_n \neq 0$ for unobservable mode)

$\Rightarrow$ adding an actuator (column in B matrix) can recover controllability (if $w_n^T b_{new} \neq 0$ for uncontrollable mode n).