16.31 Fall 2005, Homework 1

$\boxed{1}$ u is input for both systems $\Rightarrow B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

y is sum of subsystem outputs $\Rightarrow C = [C_1 \ \ C_2]$
$D = [D_1 + D_2]$

final system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [C_1 \ \ C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [D_1 + D_2] u$$

check: dimensions match ✓

$\boxed{2}$ from figure 2.17:

$$u = u_1, \quad u_2 = y_1, \quad y = y_2$$

$u_2 = y \Rightarrow \quad u_2 = C_1 x_1 + D_1 u_1$

$$\therefore \ \dot{x}_2 = A_2 x_2 + B_2 (C_1 x_1 + D_1 u_1)$$

Substitute and stack into Matrices

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u$$

$$y = [0 \ \ C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 u$$

note that if original systems are controllable and observable, so is the series interconnection

$\boxed{3}$ From Figure 2.18:

$$y = y_1 \quad , \quad u_1 = u - y_2 \, , \quad u_2 = y_1$$

The trick to this problem is to solve for y first, to get the effect of the "algebraic feedback loop" out of the way:

$$\begin{aligned} y &= C x_1 + D_1 u_1 \\ &= C x_1 + D_1 (u - y_2) \\ &= C x_1 + D_1 u - D_1 (C_2 x_2 + D_2 y) \end{aligned}$$

This is what happens when you have algebraic feedback! Solve for y:

$$(I + D_1 D_2) y = C x_1 + D_1 u - D_1 C_2 x_2$$

$$\begin{aligned} y &= (I + D_1 D_2)^{-1} \left[C x_1 + D_1 u - D_1 C_2 x_2 \right] \\ &\triangleq \Delta^{-1} C x_1 + \Delta^{-1} D_1 u - \Delta^{-1} D_1 C_2 x_2 \end{aligned}$$

Now that we have y, we can proceed.

$$\dot{x}_1 = A_1 x_1 + B_1 u - B_1 (C_2 x_2 + D_2 y)$$

$$\dot{x}_2 = A_2 x_2 + B_2 y$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 D_2 \Delta^{-1} C & -B_1 C_2 + B_1 D_2 \Delta^{-1} D_1 C_2 \\ B_2 \Delta^{-1} C & A_2 - B_2 \Delta^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 - B_1 D_2 \Delta^{-1} D_1 \\ B_2 \Delta^{-1} D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \Delta^{-1} C & \Delta^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \Delta^{-1} D_1 u$$

$\boxed{4}$ substituting into given eqns:
(a)

$$\frac{d}{dt}(x^* + \Delta x) = f(x^* + \Delta x, u^* + \Delta u)$$

Using first-order Taylor series:

$$\dot{x}^* + \Delta\dot{x} = f(x^*, u^*) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^*\\u=u^*}} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x^*\\u=u^*}} \Delta u$$

since the nominal traj satisfies:

$$\dot{x}^* \triangleq f(x^*, u^*)$$

we have

$$\boxed{\Delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_* \Delta x + \left.\frac{\partial f}{\partial u}\right|_* \Delta u}$$

and, by similar arguments

$$\boxed{\Delta y = \left.\frac{\partial h}{\partial x}\right|_* \Delta x + \left.\frac{\partial h}{\partial u}\right|_* \Delta u}$$

note that the derivative of a vector w/ respect to a scalar is

$$\frac{\partial f}{\partial a} = \begin{bmatrix} \partial f_1/\partial a \\ \partial f_2/\partial a \\ \vdots \end{bmatrix}$$ by definition...

to maintain consistency of the vector-matrix operations, we define the derivative of a scalar w/ respect to a vector as a <u>row vector</u>:

$$\frac{\partial a}{\partial x} = \left[\frac{\partial a}{\partial x_1} \quad \frac{\partial a}{\partial x_2} \quad \cdots \right]$$

Combining these $\frac{\partial(\text{vector})}{\partial(\text{vector})}$ is a matrix, in this case the <u>Jacobian</u>

$\boxed{4}$
(b)

Computed trajectory — see attached. use rk45 or Simulink

Linearized system

$$\Delta \dot{x}_1 = \Delta x_2$$

$$\Delta \dot{x}_2 = -\Delta x_1 - 3(x_2^*)^2 \Delta x_2 + x_2^* \Delta u + u^* \Delta x_2$$

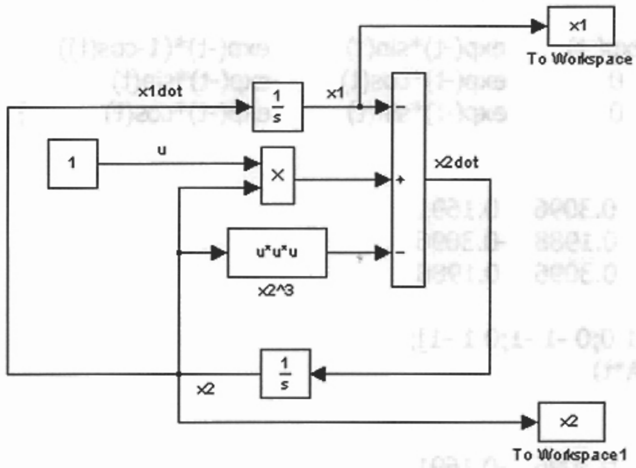
$$\Delta y = \Delta x_1$$

or

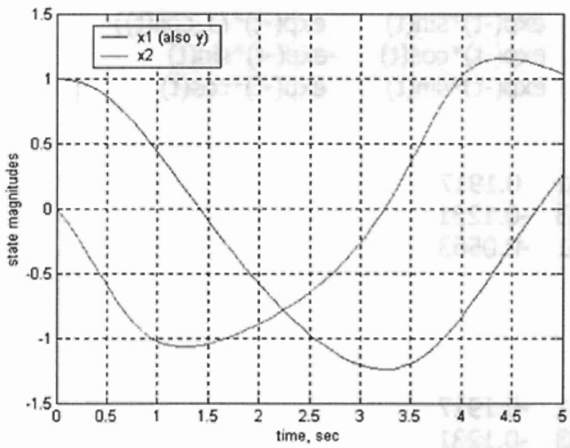
$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -3x_2^{*2} + u^* \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^* \end{bmatrix} \Delta u$$

Some of the coefficients of this system are time-varying. Plots of these coefficients are attached.

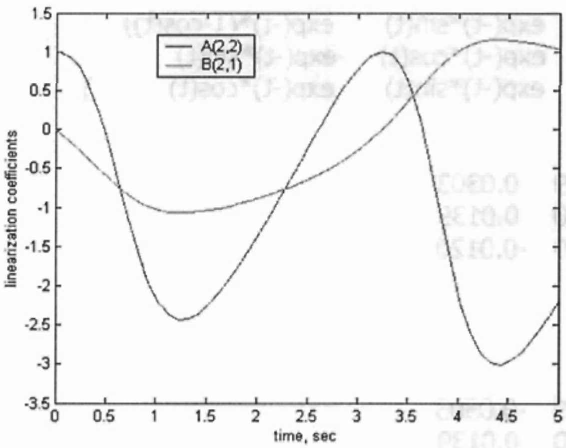
Problem 2, Part (b)



Here's the Simulink block diagram I used to compute the response – check it!



Here's the trajectories of x1 (which is y) and x2.



Here's the time histories of the coefficients of the A and B matrices
(the ones that aren't constant with time)

Problem 5, Parts (b) and (c)

```
>> t=1;
>> eAt=[exp(-t)     exp(-t)*sin(t)      exp(-t)*(1-cos(t))
            0       exp(-t)*cos(t)     -exp(-t)*sin(t)
            0       exp(-t)*sin(t)      exp(-t)*cos(t)      ]

eAt =
   0.3679    0.3096    0.1691
        0    0.1988   -0.3096
        0    0.3096    0.1988

>> A=[-1 1 0;0 -1 -1;0 1 -1];
>> expm(A*t)

ans =
   0.3679    0.3096   -0.1691
        0    0.1988   -0.3096
        0    0.3096    0.1988
```

```
>> t=2;
>> eAt=[exp(-t)     exp(-t)*sin(t)      exp(-t)*(1-cos(t))
            0       exp(-t)*cos(t)     -exp(-t)*sin(t)
            0       exp(-t)*sin(t)      exp(-t)*cos(t)      ]

eAt =
   0.1353    0.1231    0.1917
        0   -0.0563   -0.1231
        0    0.1231   -0.0563

>> expm(A*t)

ans =
   0.1353    0.1231   -0.1917
        0   -0.0563   -0.1231
        0    0.1231   -0.0563
```

```
>> t=4;
>> eAt=[exp(-t)     exp(-t)*sin(t)      exp(-t)*(1-cos(t))
            0       exp(-t)*cos(t)     -exp(-t)*sin(t)
            0       exp(-t)*sin(t)      exp(-t)*cos(t)      ]

eAt =
   0.0183   -0.0139    0.0303
        0   -0.0120    0.0139
        0   -0.0139   -0.0120

>> expm(A*t)

ans =
   0.0183   -0.0139   -0.0303
        0   -0.0120    0.0139
        0   -0.0139   -0.0120
```

$\boxed{5}$ (a) $\ddot{r} = r\dot{\theta} - \frac{k}{r^2} + u_1$

$$\ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta} + \frac{1}{r}u_2$$

(b)

$$\begin{aligned}
\dot{r}_1 &= r_2 \\
\dot{r}_2 &= r_1\theta_2 - \frac{k}{r_1^2} + u_1 \\
\dot{\theta}_1 &= \theta_2 \\
\dot{\theta}_2 &= -2\frac{r_2}{r_1}\theta_2 + \frac{1}{r_1}u_2
\end{aligned}$$

(c) if $u_1 = u_2 = 0$ + $k \neq 0$,

and we see that $r_1 = 0$ is not allowable,

we have

$$\dot{r}_2 = r_1\theta_2 - \frac{k}{r_1^2} = 0 \Rightarrow \theta_2 = \frac{k}{r_1^3}$$

$\Rightarrow \dot{\theta}_1 = \theta_2 \neq 0$ No equilibrium for these states

(d)

$x_1 = r_1 - \sigma$ $\quad x_2 = r_2 = \dot{r}_1 = \dot{x}_1 + \cancel{\dot{\sigma}}^{\,0}$

$x_3 = \theta_1 - \omega t$ $\quad x_4 = \theta_2 - \omega$

$= \dot{\theta}_1 - \omega = \dot{x}_3$ ✓

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (x_1+\sigma)(x_4+\omega) - \frac{k}{(x_1+\sigma)^2} + u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -2\frac{x_2}{(x_1+\sigma)}(x_4+\omega) + \frac{1}{(x_1+\sigma)}u_2
\end{aligned}$$

(e) if $u_1 = u_2 = 0$:

$$\dot{x}_2 = (x_1+\sigma)(x_4+\omega) - \frac{k}{(x_1+\sigma)^2} = 0$$

$$\Rightarrow (x_4+\omega) = \frac{k}{(x_1+\sigma)^3}$$

if $x_1 = 0$, $x_4 = 0$, then $\omega = \frac{k}{\sigma^3} \triangleq \frac{\sigma^3\omega^2}{\omega}$ ✓

$\dot{x}_1 = x_2 = 0$ (assumed)

$\dot{x}_2 = 0$ (from above)

$\dot{x}_3 = x_4 = 0$ (assumed)

$\dot{x}_4 = -2\frac{0}{\sigma}(x_4+\omega) = 0$ ✓

(I.o.w.'s this is how ω's dependence on σ, k is found)

$\Rightarrow [x] = 0$ is a valid equilibrium.

Linearize (drop Δ's, too messy otherwise!)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \omega x_1 + \sigma x_4 + 2\frac{k}{\sigma^3}x_1 + u_1$$

$$= (\omega + 2\omega^2)x_1 + \sigma x_4$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -2\frac{\omega}{\sigma}x_2 - \frac{1}{\sigma}u_2$$