16.31 Fall 2005, Homework 1

[1] $u$ is input for both systems $\Rightarrow B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

$y$ is sum of subsystem outputs $\Rightarrow C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $D = [D_1 + D_2]$

final system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [D_1 + D_2] u$$

checks dimensions match ✓

$$y = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 D_1 u_1$$

[2] from figure 2.17:

$u = u_1$, $u_2 = y_1$, $y = y_2$

$u_2 = y \Rightarrow u_2 = C_1 x_1 + D_1 u_1$

$\therefore \dot{x}_2 = A_2 x_2 + B_2 (C_1 x_1 + D_1 u_1)$

Substitute and stack into matrices

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 u$$

note that if original systems are controllable and observable, so is the series interconnection.

[3] From Figure 2.18:

$y = y_1$, $u_1 = u - y_2$, $u_2 = y$

The trick to this problem is to solve for $y$ first, to get the effect of the "algebraic feedback loop" out of the way:

$$y = C_1 x_1 + D_1 u_1$$
$$= C_1 x_1 + D_1 (u - y_2)$$
$$= C_1 x_1 + D_1 u - D_1 (C_2 x_2 + D_2 y)$$

This is what happens when you have algebraic feedback! Solve for $y$:

$$(I + D_1 D_2) y = C_1 x_1 + D_1 u - D_1 C_2 x_2$$
$$y = (I + D_1 D_2)^{-1} [C_1 x_1 + D_1 u - D_1 C_2 x_2]$$
$$\triangleq \Delta^{-1} C_1 x_1 + \Delta^{-1} D_1 u + \Delta^{-1} D_1 C_2 x_2$$

Now that we have $y$, we can proceed.

$$\dot{x}_1 = A_1 x_1 + B_1 u - B_1 (C_2 x_2 + D_2 y)$$
$$\dot{x}_2 = A_2 x_2 + B_2 y$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 D_2 \Delta^{-1} C & -B_1 C_2 + B_1 D_2 \Delta^{-1} D_1 C_2 \\ B_2 \Delta^{-1} C & A_2 - B_2 \Delta^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 - B_1 D_2 \Delta^{-1} D_1 \\ B_2 \Delta^{-1} D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \Delta^{-1} C_1 & -\Delta^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \Delta^{-1} D_1 u$$

**4 (a)** substituting into given eqns:

$$\frac{d}{dt}(x^* + \Delta x) = f(x^* + \Delta x, u^* + \Delta u)$$

using first-order Taylor series:

$$\dot{x}^* + \Delta\dot{x} = f(x^*, u^*) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^*\\u=u^*}} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x^*\\u=u^*}} \Delta u$$

since the nominal traj satisfies:

$$\dot{x}^* \triangleq f(x^*, u^*)$$

we have

$$\boxed{\Delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{*} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{*} \Delta u}$$

and, by similar arguments

$$\boxed{\Delta y = \left.\frac{\partial h}{\partial x}\right|_{*} \Delta x + \left.\frac{\partial h}{\partial u}\right|_{*} \Delta u}$$

note that the derivative of a vector w/ respect to a scalar is

$$\frac{\partial f}{\partial a} = \begin{bmatrix} \partial f_1/\partial a \\ \partial f_2/\partial a \\ \vdots \end{bmatrix} \quad \text{by definition...}$$

to maintain consistency of the vector-matrix operations, we define the derivative of a scalar w/ respect to a vector as a row vector:

$$\frac{\partial a}{\partial x} = \left[ \frac{\partial a}{\partial x_1} \ \ \frac{\partial a}{\partial x_2} \ \cdots \right]$$

Combining these $\frac{\partial(\text{vector})}{\partial(\text{vector})}$ is a matrix, in this ~~case~~ the Jacobian

**4 (b)**

Computed trajectory — see attached. use rk45 or Simulink

Linearized system

$$\Delta\dot{x}_1 = \Delta x_2$$

$$\Delta\dot{x}_2 = -\Delta x_1 - 3(x_2^*)^2 \Delta x_2 + x_2^* \Delta u + u^* \Delta x_2$$

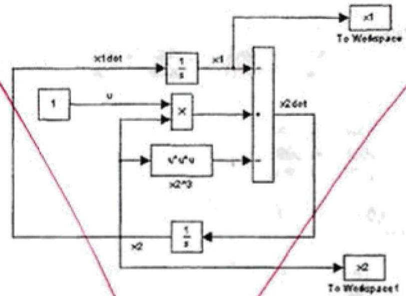
$$\Delta y = \Delta x_1$$

or

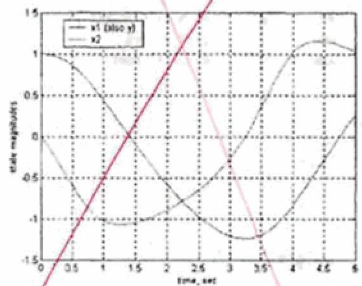
$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3x_2^{*2} + u^* \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^* \end{bmatrix} \Delta u$$

Some of the coefficients of this system are time-varying. Plots of these coefficients are attached.

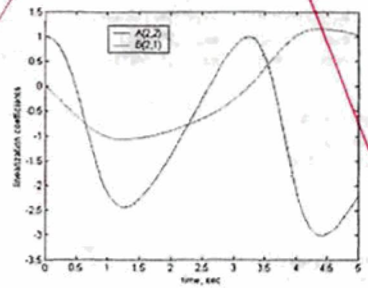
Problem 2, Part (b)




Here's the Simulink block diagram I used to compute the response – check it!




Here's the trajectories of x1 (which is y) and x2.




Here's the time histories of the coefficients of the A and B matrices (the ones that aren't constant with time)

Problem 5, Parts (b) and (c)

```
>> t=1;
>> eAt=[exp(-t)    exp(-t)*sin(t)    exp(-t)*(1-cos(t))
          0        exp(-t)*cos(t)   -exp(-t)*sin(t)
          0        exp(-t)*sin(t)    exp(-t)*cos(t)      ]

eAt =
   0.3679   0.3096   0.1691
        0   0.1988  -0.3096
        0   0.3096   0.1988

>> A=[-1 1 0;0 -1 -1;0 1 -1];
>> expm(A*t)

ans =
   0.3679   0.3096  -0.1691
        0   0.1988  -0.3096
        0   0.3096   0.1988
```

```
>> t=2;
>> eAt=[exp(-t)    exp(-t)*sin(t)    exp(-t)*(1-cos(t))
          0        exp(-t)*cos(t)   -exp(-t)*sin(t)
          0        exp(-t)*sin(t)    exp(-t)*cos(t)      ]

eAt =
   0.1353   0.1231   0.1917
        0  -0.0563  -0.1231
        0   0.1231  -0.0563

>> expm(A*t)

ans =
   0.1353   0.1231  -0.1917
        0  -0.0563  -0.1231
        0   0.1231  -0.0563
```

```
>> t=4;
>> eAt=[exp(-t)    exp(-t)*sin(t)    exp(-t)*(1-cos(t))
          0        exp(-t)*cos(t)   -exp(-t)*sin(t)
          0        exp(-t)*sin(t)    exp(-t)*cos(t)      ]

eAt =
   0.0183  -0.0139   0.0303
        0  -0.0120   0.0139
        0  -0.0139  -0.0120

>> expm(A*t)

ans =
   0.0183  -0.0139  -0.0303
        0  -0.0120   0.0139
        0  -0.0139  -0.0120
```

5 (a) $\ddot{r} = r\dot{\theta} - \frac{k}{r^2} + u_1$ or $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$

$\ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta} + \frac{1}{r}u_2$

(b) $\dot{r}_1 = r_2$

$\dot{r}_2 = r_1\theta_2 - \frac{k}{r_1^2} + u_1$ or $\dot{r}_2 = r_1\theta_2^2 - \frac{k}{r_1^2} + u_1$

$\dot{\theta}_1 = \theta_2$

$\dot{\theta}_2 = -2\frac{r_2}{r_1}\theta_2 + \frac{1}{r_1}u_2$

(c) if $u_1 = u_2 = 0$ + $k \neq 0$,

and we see that $r_1 = 0$ is not allowable,

we have

$\dot{r}_2 = r_1\theta_2 - \frac{k}{r_1^2} = 0 \Rightarrow \theta_2 = \frac{k}{r_1^3}$ or $\dot{r}_2 = 0 \Rightarrow \theta_2^2 = \frac{k}{r_1^3}$

$\Rightarrow \dot{\theta}_1 = \theta_2 \neq 0$ No equilibrium for these states.

(d) $x_1 = r_1 - \sigma$ $\quad x_2 = r_2 = \dot{r}_1 = \dot{x}_1 + \dot{\sigma}$ (→0)

$x_3 = \theta_1 - \omega t$ $\quad x_4 = \theta_2 - \omega = \dot{\theta}_1 - \omega = \dot{x}_3$ ✓

$\dot{x}_1 = x_2$

$\dot{x}_2 = (x_1+\sigma)(x_4+\omega) - \frac{k}{(x_1+\sigma)^2} + u_1$ or $\dot{x}_2 = (x_1+\sigma)(x_4+\omega)^2 - \frac{k}{(x_1+\sigma)^2} + u_1$

$\dot{x}_3 = x_4$

$\dot{x}_4 = -2\frac{x_2}{(x_1+\sigma)}(x_4+\omega) + \frac{1}{(x_1+\sigma)}u_2$

(e) if $u_1 = u_2 = 0$:

$\dot{x}_2 = (x_1+\sigma)(x_4+\omega) - \frac{k}{(x_1+\sigma)^2} = 0$

$\Rightarrow (x_4 + \omega) = \frac{k}{(x_1+\sigma)^3}$

if $x_1 = 0$, $x_4 = 0$, then $\omega = \frac{k}{\sigma^3} \triangleq \frac{\sigma^3\omega^2}{\omega}$ ✓

(I.e. w's this is how ω's dependence on σ, k is found)

$\dot{x}_1 = x_2 = 0$ (assumed)

$\dot{x}_2 = 0$ (from above)

$\dot{x}_3 = x_4 = 0$ (assumed)

$\dot{x}_4 = -2\frac{0}{\sigma}(x_4+\omega) = 0$ ✓

$\Rightarrow [x] = 0$ is a valid equilibrium.

Linearize (drop Δ's, too messy otherwise!)

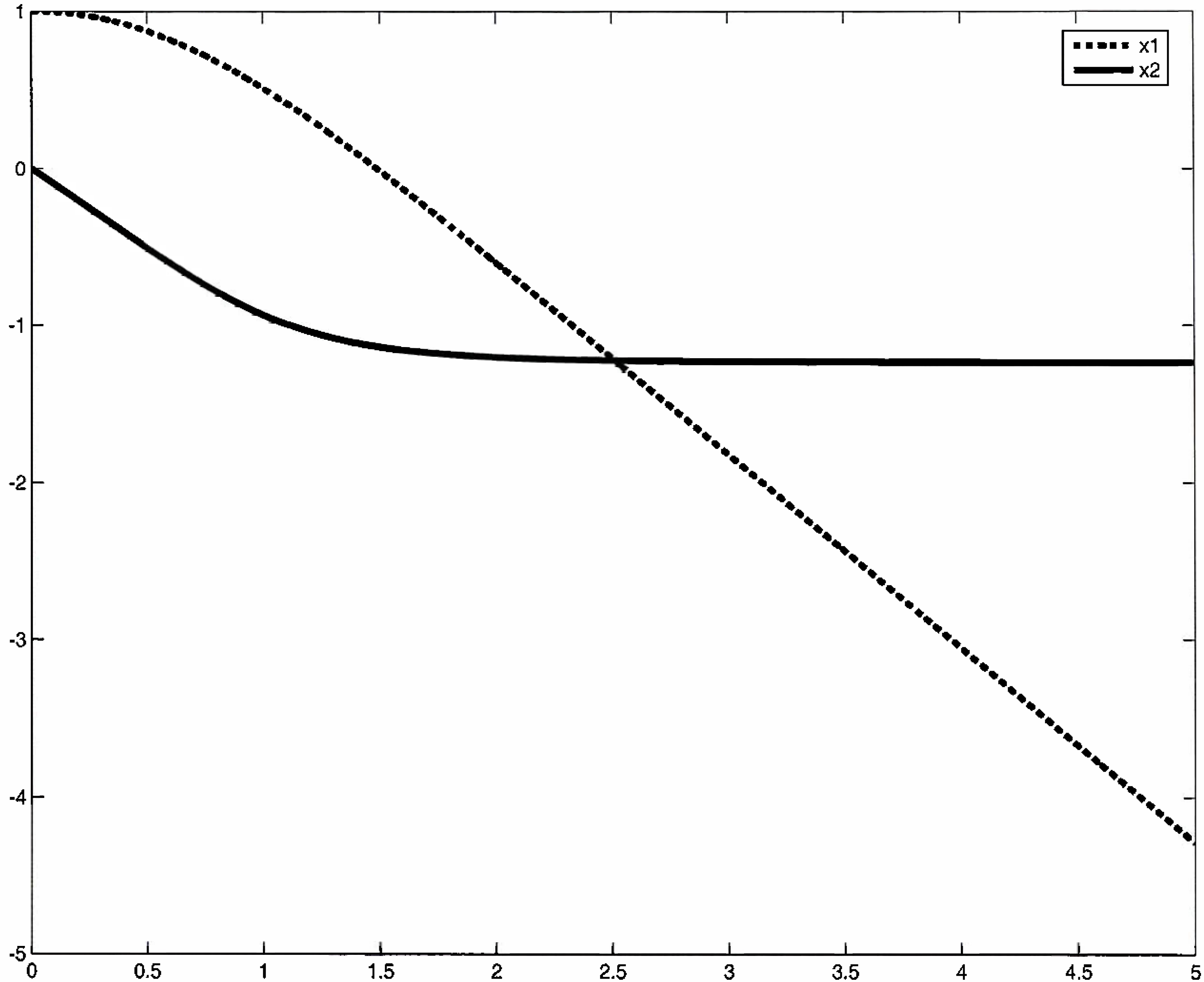
$\dot{x}_1 = x_2$

$\dot{x}_2 = \omega x_1 + \sigma x_4 + 2\frac{k}{\sigma^3}x_1 + u_1$

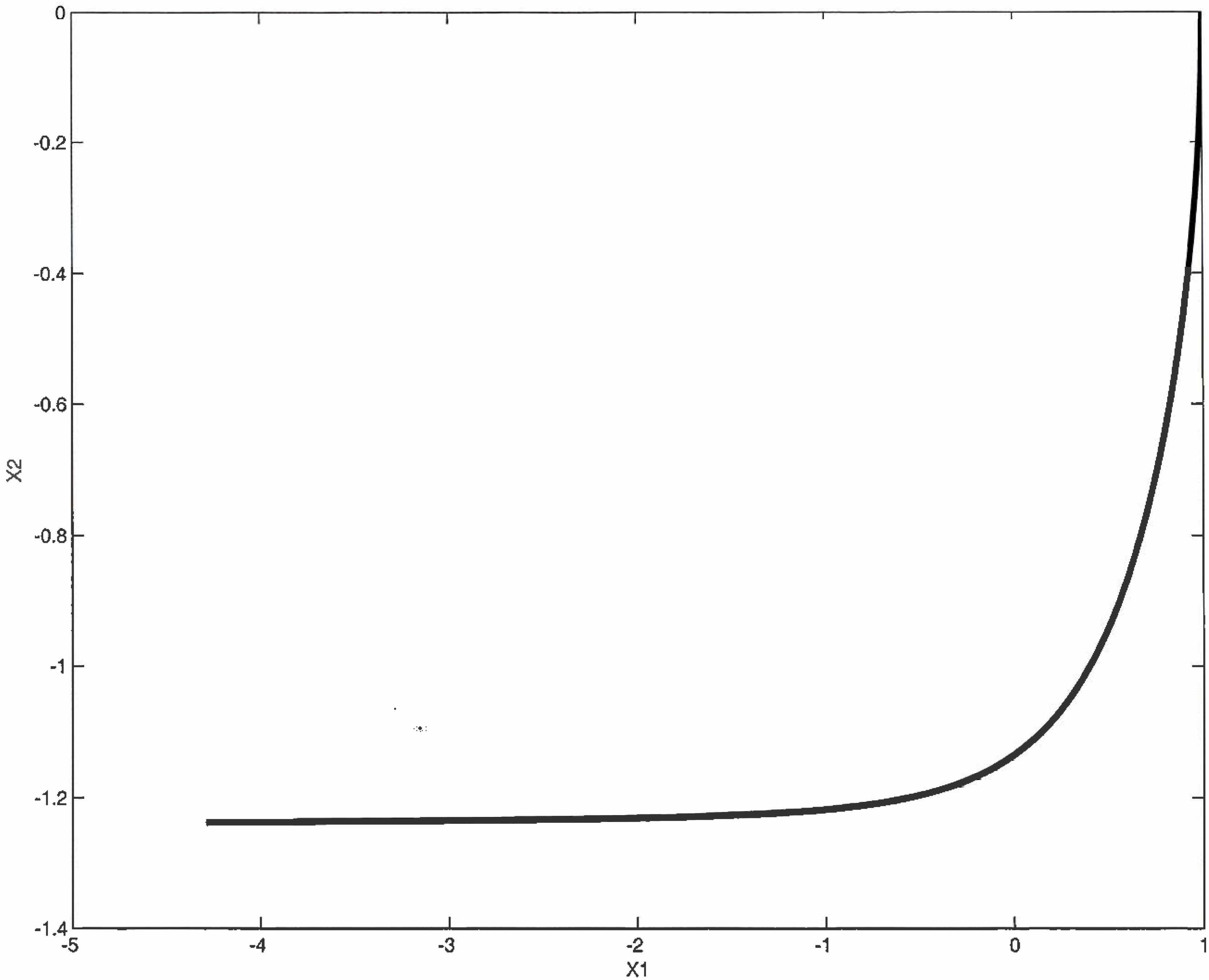
$= (\omega + 2\omega^2)x_1 + \sigma x_4$ or $= 3\omega^2 x_1 + 2\omega\sigma x_4$

$\dot{x}_3 = x_4$

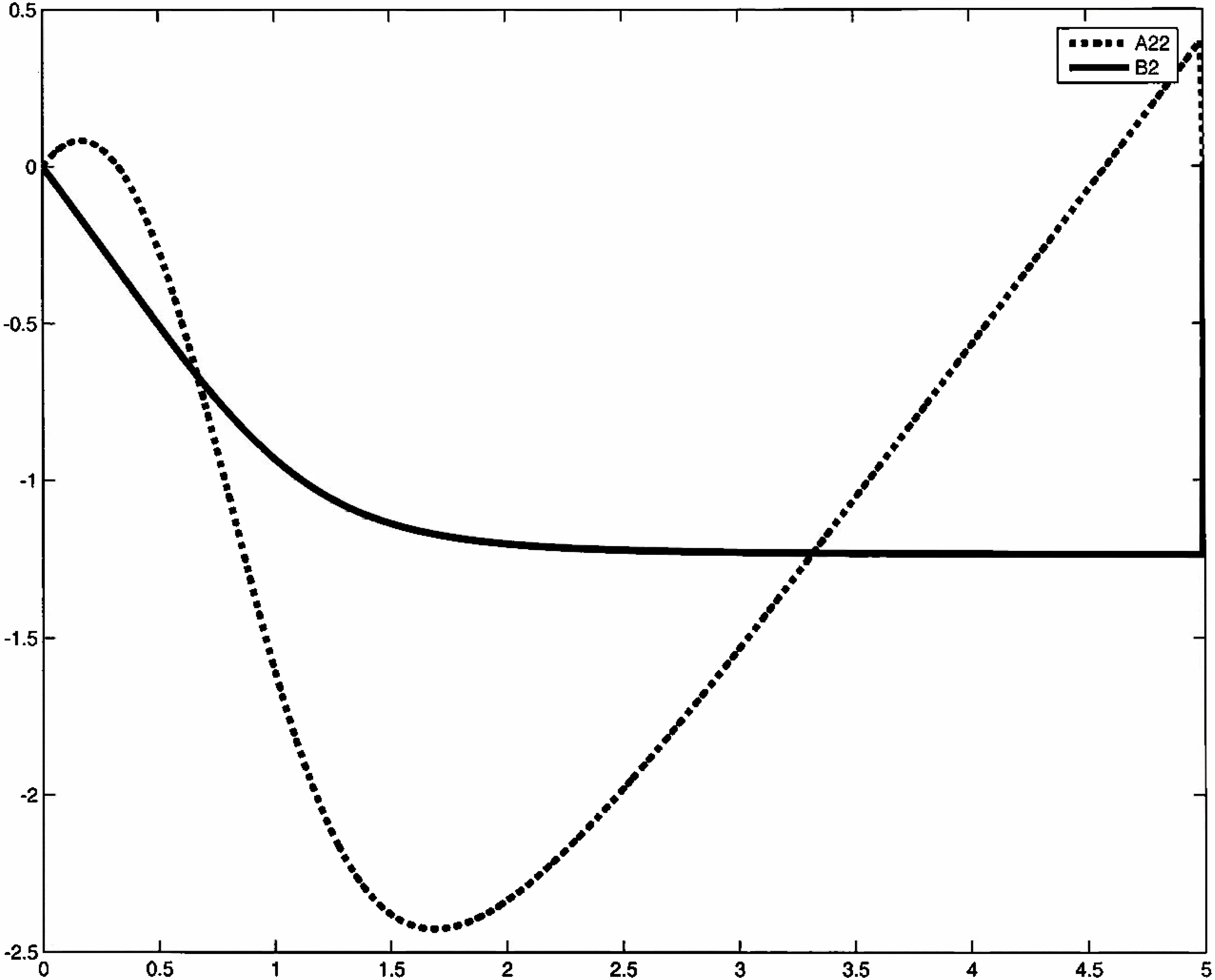
$\dot{x}_4 = -2\frac{\omega}{\sigma}x_2 \ominus \frac{1}{\sigma}u_2$ (→ +)



x1
x2
1
0
-1
-2
-3
-4
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
X2
-5
-4
-3
-2
-1
0
1
X1


A22
B2
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5