16.31 Fall 2005, Homework 2

$\boxed{1}$ $\boxed{\text{Linearity}}$ $\qquad \sum a_n y^{(n)} = \sum b_m u^{(m)}$

let $\quad y = c_1 y_1 + c_2 y_2$

$\quad u = c_1 u_1 + c_2 u_2$

$$\sum a_n (c_1 y_1 + c_2 y_2)^{(n)} \overset{?}{=} \sum b_m (c_1 u_1 + c_2 u_2)^{(m)}$$

$$c_1 \sum a_n y_1^{(n)} + c_2 \sum a_n y_2^{(n)} \overset{?}{=} c_1 \sum b_m u_1^{(m)} + c_2 \sum b_m u_2^{(m)}$$

since $\quad \sum a_n y_1^{(n)} = \sum b_m u_1^{(m)}$

and $\quad \sum a_n y_2^{(n)} = \sum b_n u_2^{(m)}$

the test equality is satisfied.

$\boxed{\text{Time-Invariance}}$

$$\sum a_n y(t-\lambda)^{(n)} \overset{?}{=} \sum b_m u(t-\lambda)^{(m)}$$

Since the derivative operation is time-invariant, this equation is satisfied.

$\boxed{\text{Non-Anticipation}}$

Taking the Laplace Xform,

$$y(s) = \frac{\sum b_m s^m}{\sum a_m s^n} u(s)$$

if $m \leq n$, the system is causal

$\boxed{2}$ (b) $y = tu'$

$L(u) = tu'$

$u = c_1 u_1 + c_2 u_2$

$L(u) = t(c_1 u_1' + c_2 u_2')$

$= c_1 t u_1' + c_2 t u_2'$

$= c_1 L(u_1) + c_2 L(u_2)$

$\therefore$ $L(u)$ is Linear

$L(u(t-\lambda)) = t\,u'(t-\lambda)$

$y(t-\lambda) = L(u)\big|_{t-\lambda} = (t-\lambda)u'(t-\lambda)$

$\therefore$ $L(u)$ is NOT Time-invariant

(d) $y(t) = u \cdot u' + 1$

$L(u) = u \cdot u'$

$u = c_1 u_1 + c_2 u_2$

$L(u) = (c_1 u_1 + c_2 u_2)(c_1 u_1' + c_2 u_2') + 1$

$= c_1^2 u_1 u_1' + c_2^2 u_2 u_2'$

$+ c_1 c_2 u_1 u_2' + c_1 c_2 u_2 u_1' + 1$

$c_1 L(u_1) + c_2 L(u_2) =$
$c_1 u_1 u_1' + c_2 u_2 u_2' + 2$

$\neq c_1 u_1 u_1' + c_2 u_2 u_2' + 2$

$\therefore$ $L(u)$ is NOT Linear

$L(u(t-\lambda)) = u(t-\lambda)\,u'(t-\lambda) + 1$

$y(t-\lambda) = L(u)\big|_{t-\lambda} = u(t-\lambda)\,u'(t-\lambda) + 1$

$\therefore$ $L(u)$ is Time-Invariant

(f) $y(n) = u(n-1)u(n+1)$

$L(u) = u(n-1)u(n+1)$

$u = c_1 u_1 + c_2 u_2$

$$L(u) = [c_1 u_1(n-1) + c_2 u_2(n-1)][c_1 u_1(n+1) + c_2 u_2(n+1)]$$

$$= c_1^2 u_1(n-1)\, u_1(n+1)$$

$$+ c_2^2 u_2(n-1)\, u_1(n+1) + \text{CROSS-TERMS}$$

$$\neq c_1 u_1(n-1) u_1(n+1) + c_2 u_2(n-1) u_2(n+1)$$

$\therefore$ $L(u)$ is <u>NOT</u> Linear

$$L(u(n-\lambda)) = u(n-\lambda-1)\, u(n-\lambda+1)$$

$$Lu\big|_{n-\lambda} = u(n-\lambda-1)\, u(n-\lambda+1)$$

$\therefore$ $L(u)$ is <u>Time-Invariant</u>

(h) $y(n) = \left(\frac{1}{2}\right)^n u(n)$

$$L(c_1 u_1 + c_2 u_2) = \left(\tfrac{1}{2}\right)^n (c_1 u_1 + c_2 u_2)$$

$$= c_1 \left(\tfrac{1}{2}\right)^n u_1 + c_2 \left(\tfrac{1}{2}\right)^n u_2$$

$\therefore$ $L(u)$ is <u>Linear</u>

$$L(u(n-\lambda)) = \left(\tfrac{1}{2}\right)^n u(n-\lambda)$$

$$\neq \left(\tfrac{1}{2}\right)^{n-\lambda} u(n-\lambda)$$

$\therefore$ $L(u)$ is <u>NOT</u> Time-Invariant

$\boxed{3}$ $sx_3 = x_3 + u \Rightarrow x_3 = \frac{1}{s-1} u$

$sx_2 = x_2 + x_3 \Rightarrow x_2 = \frac{1}{s-1} x_3$

$sx_1 = x_1 + x_2 \Rightarrow x_1 = \frac{1}{s-1} x_2$

$$y(s) = x_1 + 3x_3$$

$$= \frac{1}{s-1} x_2 + 3\frac{1}{s-1} u$$

$(s^2-2s+1)s-1$

s^3-3s^2+3s-1

$$= \frac{1}{s-1}\left(\frac{1}{s-1}\left(\frac{1}{s-1}u\right)\right) + 3\frac{1}{s-1}u$$

$$= \left[\frac{1}{(s-1)^3} + \frac{3(s-1)^2}{(s-1)^3}\right]u$$

$$= \frac{3s^2-6s+4}{(s-1)^3} u$$

$\boxed{4}$ $sx_1 = -2x_1 \Rightarrow (s-2)x_1 = 0$

$sx_2 = -x_1 - 4x_2 \Rightarrow x_2 = \frac{-1}{s+4} x_1$

$sx_3 = -x_1 - x_2 - 5x_3 + u \Rightarrow (s+5)x_3 = \left(-1 + \frac{1}{s+4}\right)x_1 + u$

eliminate x_1:

$$(s+2)x_1 = -\frac{s+4}{s+3}\left[(s+5)x_3 - u\right](s+2) = 0$$

$$x_3 = \left(\frac{1}{s+5}\right)u$$

modes associated w/ x_1 & x_2 are <u>uncontrollable</u>

(first to rows of controllability matrix are zero)

$\boxed{5}$ (a) $y'' + 3y' + 10y = 2u$

let $\left.\begin{array}{l} x_1 = y \\ x_2 = y' \end{array}\right\}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) $y'' + 10y = 2u$

same method

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}$$

(c) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \; ; \; y = x_1$$

(d) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \; ; \; y = x_1$$

$y \quad y' \quad y''$

$\boxed{6}$ $y''' - 3y'' + 3y' - y = 3u'' - 6u' + 4u$

$\Rightarrow$ by inspection of #3 soln

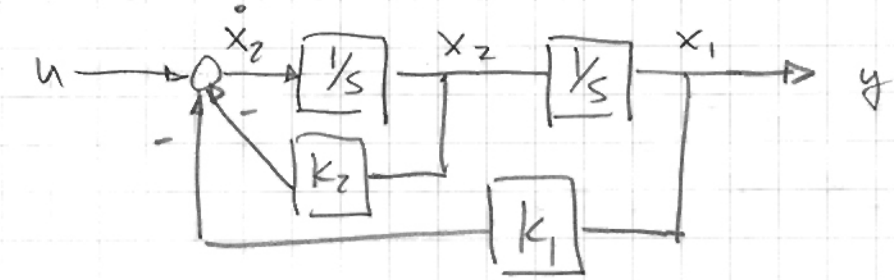
$\boxed{7}$ (a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u \; ; \; y = x_1$$

(b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} u \; ; \; y = x_1$$

(c) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \; ; \; y = x_1$$

(d) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \; ; \quad y = x_1$

$\boxed{8}$



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u - k_2 x_2 - k_1 x_1$$

$$y = x_1 \;, \quad y' = x_2 \;, \quad y'' = u - k_2 y' - k_1 y$$

$$\underbrace{(s^2 + k_2 s + k_1)}_{} y = u$$

roots of the characteristic polynomial are

$$\frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2} = a \pm bj$$

$\boxed{\text{if } b \neq 0}$ $\quad a = -k_2/2 \quad \Rightarrow \quad \underline{\underline{k_2 = -2a}}$

$b = \frac{1}{2}\sqrt{4k_1 - k_2^2}$

$4b^2 = 4k_1 - 4a^2 \qquad \underline{\underline{k_1 = a^2 + b^2}}$

$\boxed{\text{if } b = 0}$ $\quad (s-a)(s-a)\, y = u$

$$(s^2 - 2a + a^2) y = 0$$

$$\underline{\underline{k_1 = a^2}}$$
$$\underline{\underline{k_2 = -2a}}$$

$\boxed{9}$

$$x_1 = y$$

$$\dot{x}_1 = x_2 = \dot{y}$$

$$\dot{x}_2 = \ddot{y} = -2\zeta\omega_n\dot{y} - \omega_n^2 y$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$