16.31 Fall 2005, Homework 4

1

$$\dot{x}_1 = x_2 + u$$
$$\dot{x}_2 = x_1 - u$$

$$\ddot{x}_2 = \dot{x}_1 - \dot{u} = (x_2 + u) - \dot{u}$$

$$x_2 = \frac{1-s}{s^2-1} u = \frac{-1}{s+1} u$$

$$s x_1 = x_2 + u = \left(\frac{-1}{s+1} + 1 \right) u = \frac{s}{s+1} u$$

$$y = x_1 = \frac{1}{s} \frac{s}{s+1} u \Rightarrow \boxed{y = \frac{1}{s+1} u}$$

2

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \right\} \text{First "Block"}$$

$$y_1 = \frac{1}{s^2+3s+2} u \quad \text{(by analogy to HW \#2)}$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$y_2 = \frac{1}{s^2+s} u \quad \text{(by analogy to HW \#2)}$$

$$y = y_1 + y_2 = \left(\frac{1}{s^2+3s+2} + \frac{1}{s^2+s} \right) u$$

$$= \frac{s^2+4s+2}{s(s+1)^2(s+2)} u$$

$\boxed{3}$ All proofs are for distinct eigenvalues.

(a) $$A^m = (M\Lambda M^{-1})^m$$
$$= M\Lambda M^{-1} M\Lambda M^{-1} \cdots M\Lambda M^{-1}$$
$$= M\Lambda^m M^{-1}$$

$\therefore$ eigenvalue matrix for A^m is Λ^m, eigenvectors are in M. diagonal element of Λ^m are

$\lambda(A^m) = \{\lambda_1^m, \lambda_2^m \cdots\}$ ✓

(b) $$A^{-1} = (M\Lambda M^{-1})^{-1}$$
$$= M\Lambda^{-1}M = M\begin{bmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_n \end{bmatrix}M^{-1}$$

$\therefore \lambda(A^{-1}) = \{1/\lambda_1 \cdots 1/\lambda_n\}$ ✓

(c) $$A^T = (M\Lambda M^{-1})^T$$
$$= (M^{-1})^T \Lambda^T M^T$$

But $\Lambda^T = \Lambda$

$\therefore A^T = L^{-1}\Lambda L$ where $L = M^T$

and $\lambda(A^T) = \lambda(A)$ ✓

(d) $$kA = kM\Lambda M^{-1} = Mk\Lambda M^{-1}$$
$$= M\begin{bmatrix} k\lambda_1 & & \\ & \ddots & \\ & & k\lambda_n \end{bmatrix}M^{-1}$$

$\therefore \lambda(kA) = \{k\lambda_1, \cdots k\lambda_n\}$ ✓

4

(a) Let $(A-\lambda_i I)^k x_k = 0$ (x_k has rank k)

and $(A-\lambda_j I)^p x_p = 0$ (x_p has rank p)

be the highest-order eqns that equal zero for eigenvectors λ_i and λ_j respectively

assume $p > k$ w/out loss of generality.

If $x_k = a x_p$, and since

$$(A-\lambda_j I)^p x_p \cdot a = 0$$

we get

$$(A-\lambda_j I)^p x_k = 0$$ I.E. x_k has rank p!

This violates the original assumption, therefore

$x_k \neq a x_p \Rightarrow x_k$ & x_p are independent

(b) $$\text{eig}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}\right\} : \det\begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-1 & 0 \\ -2 & -3 & \lambda-2 \end{vmatrix} = 0$$

$$\text{eig}\{A\} = \{1, 1, 2\}$$

$$\det(\lambda I - A)\Big|_{\lambda=1} = \det\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & -1 \end{bmatrix} = 0$$

$\Rightarrow$ only one linearly independent eigenvector exists for $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ no info

$\Rightarrow x_1 = 0$

$\Rightarrow x_3 = -3x_2$

from prev page, $\underline{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

for $\lambda = 2$,

$$(\lambda I - A)\Big|_{\lambda=2} \underline{x}_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{matrix} \Rightarrow x_1 = 0 \\ \Rightarrow x_2 = 0 \\ \Rightarrow x_3 = \text{anything} \end{matrix}$$

$$\therefore \underline{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Generalized Eigenvector:</u>

$$(\lambda I - A)^2 \Big|_{\lambda=1} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

We need:

$$(\lambda I - A)\underline{x}_{1g} \neq 0$$

$$(\lambda I - A)^2 \underline{x}_{1g} = 0$$

I.O.W.'s

$$5x_1 + 3x_2 + x_3 = 0$$

$$2x_1 + 3x_2 + x_3 \neq 0$$

subtract eqns: $3x_1 \neq 0$

choose $\underline{x}_{1g} = \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix}$ to satisfy 1st eqn

(c) Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -3 & -8 & 1 \end{bmatrix}$

$M^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & 3 & 1 \end{bmatrix}$ (using Matlab)

Check $M^{-1}AM = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = J$ ✓

$\boxed{5}$ (a)

(1) $\vec{\varepsilon}_1 \to \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$

$\vec{\varepsilon}_2 \to \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$

Arbitrary vector $= \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{\varepsilon}_1 + x_2 \vec{\varepsilon}_2$

$$T(\underline{x}) = x_1 \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + x_2 \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}}$$

(2) $\det(\lambda I - T) = \begin{vmatrix} \lambda - c\theta & s\theta \\ -s\theta & \lambda - c\theta \end{vmatrix}$

$$= \lambda^2 - 2c\theta\lambda + c^2\theta + s^2\theta$$

$$= \lambda^2 - 2c\theta\lambda + 1$$

$$\lambda = \frac{2c\theta}{2} \pm \frac{\sqrt{4c^2\theta - 4}}{2}$$

$$\boxed{\lambda = \cos(\theta) \pm j \sin\theta}$$

<u>eigenvectors</u>

$$(\lambda_1 I - A)\underline{x} = \begin{bmatrix} j s\theta & -s\theta \\ s\theta & j s\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$\left.\begin{array}{ll} \text{first eqn} = & j s\theta\, x_1 - s\theta\, x_2 = 0 \\ j \times (\text{second eqn}) = & j s\theta\, x_1 - s\theta\, x_2 = 0 \end{array}\right\} \; x_2 = j x_1$$

$$\boxed{\underline{x}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}}$$

$$(\lambda_2 I - A)\underline{x} = \begin{bmatrix} -j s\theta & -s\theta \\ s\theta & -j s\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$-j s\theta\, x_1 - s\theta\, x_2 = 0 \Rightarrow x_2 = -j x_1$$

$$\boxed{\underline{x}_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}}$$

(b)

(1) $$\det(\lambda I - A) = \begin{vmatrix} \lambda - k_1 & (k_1 - k_2) \\ 0 & \lambda - k_2 \end{vmatrix}$$

$$= \lambda^2 - (k_1 + k_2)\lambda + k_1 k_2$$

$$= (\lambda - k_1)(\lambda - k_2)$$

$$\boxed{\lambda = k_1, k_2}$$

eigenvectors:

$$(\lambda_1 I - A)\underline{x} = \begin{bmatrix} 0 & (k_1-k_2) \\ 0 & (k_1-k_2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

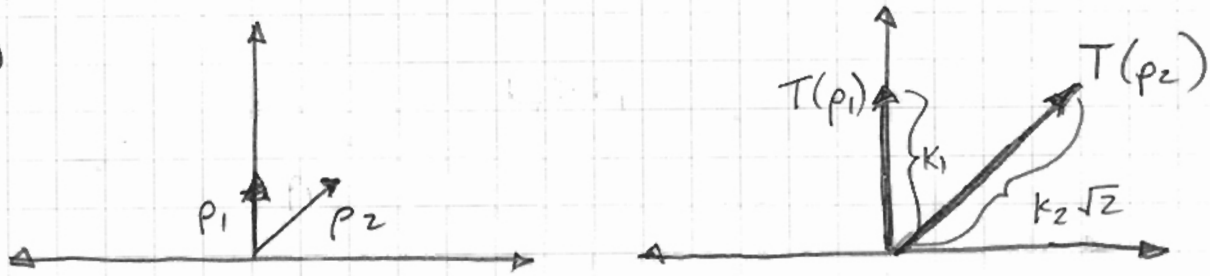
$$\Rightarrow x_2 = 0$$

$$(\lambda_2 I - A)\underline{x} = \begin{bmatrix} (k_2-k_1) & (k_1-k_2) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = x_2$$

$$\boxed{\text{Eigenvectors} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

(2)



$$T(\rho_2) = \begin{bmatrix} k_1 + k_2 - k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix}$$

(3) $k_1 = k_2 \Rightarrow T =$ identity matrix

(i) geometrically, the eigenvectors directions ρ_1 & ρ_2 become arbitrary; that is any vector ρ is transformed to $k\rho$ w/out any change of direction

(ii) eigenvalues are k and k (repeated)

eigenvectors are any arbitrary independent vectors.

$\boxed{6}$ (a) $A = \begin{bmatrix} -1 & 1/2 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = -1, 1$

Using the Cayley-Hamilton Method (pp 157 DeRusso)

$$e^{-t} = \alpha_0 - \alpha_1$$
$$e^{t} = \alpha_0 + \alpha_1$$
$$\alpha_0 = \tfrac{1}{2}(e^{t} + e^{-t})$$
$$\alpha_1 = \tfrac{1}{2}(e^{t} - e^{-t})$$

and
$$e^{At} = \alpha_0 I + \alpha_1 A$$
$$= \tfrac{1}{2}(e^{t}+e^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \tfrac{1}{2}(e^{t}-e^{-t})\begin{bmatrix} -1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t} & \frac{1}{4}(e^{t} - e^{-t}) \\ 0 & e^{t} \end{bmatrix}$$

for $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$x(t) = e^{At}x(0) = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ 2e^{t} \end{bmatrix} \checkmark$$

(b) $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad |I\lambda - A| = (\lambda+2)\lambda + 1 = 0$

$$= \lambda^2 + 2\lambda + 1 = 0$$
$$\lambda = -1, -1$$

Following Example 3.10-13

$$e^{-t} = \alpha_0 - \alpha_1$$
$$te^{-t} = \alpha_1$$
$$\Rightarrow \quad \alpha_0 = e^{-t} + te^{-t}$$

$$e^{At} = \left[e^{-t}+te^{-t}\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} - te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}+te^{-t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} te^{-t} \\ e^{-t}+te^{-t} \end{bmatrix}$$

$\boxed{7}$ $A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$ $\quad |\lambda I - A| = \lambda(\lambda+2) - 8 = 0$

$$\lambda^2 + 2\lambda - 8 = 0$$

$$(\lambda-2)(\lambda+4) = 0$$

$$\lambda = 2, -4$$

$$e^{2t} = \alpha_0 + 2\alpha_1$$

$$e^{-4t} = \alpha_0 - 4\alpha_1$$

$$\alpha_1 = \frac{1}{6}\left(e^{2t} - e^{-4t}\right)$$

$$\alpha_0 = \frac{1}{3}\left(2e^{2t} + e^{-4t}\right)$$

$$\exp(At) = \frac{1}{3}\left(2e^{2t}+e^{-4t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{6}\left(e^{2t}-e^{-4t}\right)\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}\left(2e^{2t}+e^{-4t}\right) & \frac{1}{6}\left(e^{2t}-e^{-4t}\right) \\ \frac{4}{3}\left(e^{2t}-e^{-4t}\right) & \frac{1}{3}\left(e^{2t}+2e^{-4t}\right) \end{bmatrix}$$

(cover)

$y(t) = 4x_1 + 1x_2$

$$\underline{x} = \exp(A)\, x(0)$$

$$\frac{1}{6}\begin{bmatrix} 4e2 + 2e4 & e2 - e4 \\ 8e2 - 8e4 & 2e2 + 4e^4 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6e4 \\ -24e^4 \end{bmatrix} = \begin{bmatrix} e^{-4t} \\ -4e^{-4t} \end{bmatrix}$$

$$y = 4e^{-4t} - 4e^{-4t} = 0$$

$\boxed{\text{What happened?}}$ The eigenvector associated w/ $\lambda_2 = -4$ is =

$$\begin{bmatrix} -4 & -1 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = -4x_1$$

$$\underline{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$\Rightarrow$ if IC is $x(0) = \beta\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

then $x(t) = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t}$ ✓

Also, $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \Rightarrow$ Rank is 1

$\therefore$ There is a loss of observalility.

$C\begin{bmatrix} 1 \\ -4 \end{bmatrix} = 0 \Rightarrow e^{-4t}$ mode is <u>unobservable</u>

$\boxed{8}$ $A^k = \alpha_0 I + \alpha_1 A$

(a) the eqn must also be satisfied by the eigenvalues. for $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

the eigenvalues are -1 and -1

since the eigenvalue are repeating, we must also differentiate the polynomial and substitute in the eigenvalues. This gives

$$\lambda^k = \alpha_0 + \alpha_1 \lambda$$

$$k \lambda^{k-1} = \alpha_1$$

for $\lambda = -1$ we have

$$-1^k = \alpha_0 - \alpha_1$$
$$k(-1)^{k-1} = \alpha_1$$

if k is even,

$$\alpha_1 = -k$$
$$\alpha_0 = -k + 1$$

k even

$$\Rightarrow \boxed{A^k = (1-k)I - kA}$$

if k is odd

$$\alpha_1 = k$$
$$\alpha_0 = k - 1$$

k odd

$$\Rightarrow \boxed{A^k = (k-1)I + kA}$$

$k = 0 \Rightarrow A^k = I$ ✓
$k = 1 \Rightarrow A^k = A$ ✓

check higher orders in Matlab!

$$A^k = (-1)^k \left[(1-k)I - kA \right]$$

$\boxed{18}$
(f) e^{At} where $A = \begin{bmatrix} \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} & \\ & -1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$

Since $A^k = \begin{bmatrix} A_1^k & \\ & A_2^k \end{bmatrix}$, e^{At} is also block diagonal.

$e^{A_2 t} = e^{-t}$, since it is scalar

for $A_1 = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, we have

$\lambda = \pm 3j$ and the Cayley-Hamilton Coefficients

$$e^{3jt} = \alpha_0 + 3j\alpha_1$$
$$e^{-3jt} = \alpha_0 - 3j\alpha_1$$

$e^{3jt} - e^{-3jt} = 6j\alpha_1$ $\quad \alpha_1 = \frac{1}{3}\sin 3t$

$e^{3jt} + e^{-3jt} = 2\alpha_0$ $\quad \alpha_0 = \cos 3t$

$$e^{A_1 t} = \cos 3t\, I + \frac{1}{3}\sin 3t \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos 3t & -\sin 3t & 0 \\ \sin 3t & \cos 3t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$\boxed{8}$ (g) $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \text{eig}(A) = \det(\lambda I - A)$

$= \lambda^4 = 0 \Rightarrow \lambda = 0,0,0,0$

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \alpha_3 A^3$$

$$e^{0t} = 1 = \alpha_0 + \alpha_1(0) + \alpha_2(0)^2 + \alpha_3(0)^3$$

$$te^{0t} = t = \alpha_1 + 2\alpha_2(0) + 3\alpha_3(0)^2$$

$$t^2 e^{0t} = t^2 = 2\alpha_2 + 6\alpha_3(0)$$

$$t^3 e^{0t} = t^3 = 6\alpha_3$$

$$e^{At} = I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{6}t^3A^3$$

$\boxed{9}$ $(A^k)^{-1}$ does not necessarily exist; this is easy to show by one counter-example.

If $\lambda_j(A) = 0$ and λ's are distinct then

$$A^{-1} = M\Lambda^{-1}M = M\begin{bmatrix} 1/\lambda_1 & & & & \\ & \ddots & & & \\ & & 1/0 & & \\ & & & \ddots & \\ & & & & 1/\lambda_n \end{bmatrix}M^{-1}$$

so the matrix is non-invertable

but $(e^{At})^{-1}(e^{At}) = I \Rightarrow$

$$(\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \alpha_n A^n)(\beta_0 I + \beta_0 A + \beta_0 A^2 + \cdots \beta_n A^n) = I$$

(*) $$= (\alpha_0 + \beta_0 - 1)I + (\alpha_1 + \beta_1)A + \cdots (\alpha_n + \beta_n)A^n = 0$$

$\alpha_0, \cdots \alpha_n$ can always be found, as the soln for e^{At} by Cayley-Hamilton

And if we solve for

$$\alpha_0 + \beta_0 - 1 = 1$$

$$\alpha_1 + \beta_1 = \lambda_i$$

$$\alpha_2 + \beta_2 = \lambda_i^2$$

$$\vdots$$

$$\alpha_n + \beta_n = \lambda_i^n$$

for some eigenvalue $\lambda_i \neq 0$, then (*) is $P(A) \equiv 0$ and the soln for $(e^{At})^{-1}$ can be constructed.

NOW if $\lambda_i = 0 \ \forall i$, then

$$A = MJM^{-1}$$

where $J = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$

In this case,

$$e^{At} = M^{-1}\left(I + At + A^2 t^2/2! + \cdots A^{n-1} t^{n-1}/(n-1)!\right) M$$

$$= M^{-1} \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & t & t^2/2! \\ & \emptyset & & 1 \ddots & 1 & t \\ & & & & & 1 \end{bmatrix} M$$

this matrix is invertable, since all of it's eigenvalues are equal to one!

$\boxed{10}$ We have shown that controllability require full rank of $\mathcal{C}(A, B)$. But this question is about proving that it is possible for this matrix to attain full rank. To do this we need only show that there exists an A, B pair for which $\text{rank}(\mathcal{C}) = n$. Adding more rows to $\mathcal{C}$ cannot increase the rank beyond n, so we will prove for the case

NOTE: $n+1$ columns! $\longrightarrow$ $[B \quad AB \quad \cdots \quad A^n B] = \mathcal{C}_n$

Choose A such that it's eigenvalues λ_i are distinct

Choose $B = \Sigma \alpha_k x_k$, where x_k are the eigenvectors, and $\alpha_k \neq 0 \;\; \forall k$

then

$$A^j B = A^j \Sigma \alpha_k x_k$$

$$= \Sigma \alpha_k A^j x_k = \Sigma \alpha_k \lambda_k^j x_k$$

Doing this for all the rows of $\mathcal{C}_n$,

$$\mathcal{C}_n = [\Sigma \alpha_k x_k \quad \Sigma \alpha_k \lambda_k x_k \quad \cdots \quad \Sigma \alpha_k \lambda_k^n x_k]$$

Now, consider the test for loss of rank.

if $\mathcal{C}_n \underline{k} = 0$ for $\underline{k} \neq 0$, then

$\mathcal{C}_n$ has rank $n-1$ or less.

Rewriting $\mathcal{C}_n \underline{k}$ for $\underline{k} = \begin{bmatrix} k_0 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$; (over)

$$K_0 \Sigma \alpha_k x_k + K_1 \Sigma \alpha_k \lambda_k x_k + K_2 \Sigma \alpha_k \lambda_k^2 x_k$$

(*) $$+ \cdots K_n \Sigma \alpha_k \lambda^n x_k = 0$$

we can collect terms in x_k and write this as

$$(K_0 + K_1 \lambda_1 + K_2 \lambda_1^2 + \cdots K_n \lambda_1^n) \alpha_1 x_1$$
$$+ (K_0 + K_1 \lambda_2 + K_2 \lambda_2^2 + \cdots K_n \lambda_2^n) \alpha_2 x_2$$
$$+ \cdots \qquad \alpha_n x_n = 0$$

Now, since $x_k \neq 0 \ \forall k$ (distinct λ's)

and since $\alpha_k \neq 0 \ \forall k$, we need

all the coefficients above to be $= 0$.

Re-write this in matrix notation:

(**) $$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^n \\ 1 & \lambda_2 & \lambda_2^2 & & \lambda_2^n \\ \vdots & & & & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^n \end{bmatrix} \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

But the rank of this matrix is <u>exactly</u> n,

because the solution in each row is simply the characteristic equation:

$$\det|\lambda_k I - A| = 0$$

since $\exists$ one and only one soln $\underline{K}$ that satisfies all n of these eqns, setting any $K_j = 0 \Rightarrow$ no non-zero soln to (**)

But this means (*) has no soln w/ $K_j = 0$ some single k_j

$$\Rightarrow \text{Rank}(\mathcal{C}_n) = n$$

The last step in that proof is a little weak. We need to prove that the "Vander Monde" matrix is invertable

$$\text{ie., } \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & & & & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \neq 0$$

PROOF:

① Subtract row 1 from all remaining rows:

$$= \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 0 & \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^{n-1} - \lambda_1^{n-1} \\ \vdots & & & & \\ 0 & \lambda_n - \lambda_1 & \lambda_n^2 - \lambda_1 & & \lambda_n^{n-1} - \lambda_1 \end{bmatrix}$$

② Expand about first row; factor first column out of the result

$$= \det \begin{bmatrix} \lambda_2 - \lambda_1 & & & \\ & \lambda_3 - \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_1 \end{bmatrix} \times$$

$$\det \begin{bmatrix} 1 & \lambda_2 + \lambda_1 & \lambda_2^2 + \lambda_1^2 & \cdots & \lambda_2^{n-2} + \lambda_1^{n-2} \\ 1 & \lambda_3 + \lambda_1 & \lambda_3^2 + \lambda_1^2 & \cdots & \lambda_3^{n-2} + \lambda_1^{n-2} \\ \vdots & & & & \\ 1 & \lambda_n + \lambda_1 & \lambda_n^2 + \lambda_1^2 & \cdots & \lambda_n^{n-2} + \lambda_1^{n-2} \end{bmatrix}$$

③ Repeat step 1; the result looks like the result of step 1 except λ_1 is replaced by λ_2 and the dimension has been reduced again (over)

$$= \prod_{j=2}^{n} (\lambda_j - \lambda_1) \cdot \det \begin{bmatrix} 1 & \lambda_2 + \lambda_1 & \lambda_2^2 + \lambda_1^2 & \cdots & \lambda_2^{n-2} + \lambda_1^{n-2} \\ 1 & \lambda_3 - \lambda_2 & \lambda_3^2 - \lambda_2^2 & \cdots & \lambda_3^{n-2} - \lambda_2^{n-2} \\ \vdots & & & & \\ 1 & \lambda_n - \lambda_2 & \lambda_n^2 - \lambda_2^2 & \cdots & \lambda_n^{n-2} - \lambda_2^{n-2} \end{bmatrix}$$

④ Repeat step ②:

$$= \prod_{j=2}^{n} (\lambda_j - \lambda_1) \prod_{k=3}^{n} (\lambda_k - \lambda_2) \det \begin{bmatrix} 1 & \lambda_3 + \lambda_2 & & \\ 1 & \lambda_4 + \lambda_2 & & \\ \vdots & \vdots & & \\ 1 & \vdots & \cdots & \lambda_n^{n-3} + \lambda_2^{n-3} \end{bmatrix}$$

you get the idea. Continue recursively, and the result is

$$= \prod_{j=2}^{n} (\lambda_j - \lambda_1) \prod_{k=3} (\lambda_k - \lambda_2) \prod_{l=4} (\lambda_l - \lambda_3) \cdots (\lambda_n - \lambda_{n-1})$$

This is the result we are looking for. as long as

$$\lambda_i \neq \lambda_j \quad \forall \ i \neq j \ ;$$

the determinant will be non-zero and the inverse of the VanderMonde matrix exists (NOTE: DO we need $\lambda_k \neq 0 \ \forall \ k$???)