HOMEWORK 5

PROBLEM 1 DeRusso 3.30: Start with an orthoganal basis that spans S $\Rightarrow$ any vector in S can be represented as

$$x = \sum_{i=1}^{n} \alpha_i \underline{e}_i \qquad \underline{e}_i \text{ are basis vectors.}$$

then any vector in S_1 can also be written as a combination of α's, where some α's are zero. Re-order e_i's such that all the vectors that span S_1 come first, i.e.

$$\text{if } y \in S_1 \text{ then } y = \sum_{k=1}^{m} \alpha_k \underline{e}_k$$

where $m < k$. Now $x \notin S_1$ implies that $\alpha_k, k > m \neq 0$; that is the decomposition contains basis vectors that are not part of those spanning S_1.

Now, for any x,

$$x = \sum_{k=1}^{m} \alpha_k \underline{e}_k + \sum_{j=m+1}^{n} \alpha_j \underline{e}_j$$

$$= \quad y \quad + \quad y_1$$

Since $\underline{e}_j^T \underline{e}_k = 0 \;\forall\; j \neq k$ (orthoganal basis),

$$y^T y_1 = \sum_{k=1}^{m} \sum_{j=m+1}^{n} \alpha_k \alpha_j \underline{e}_k^T \underline{e}_j = 0$$

Furthermore, for any other $x_1 \in S_1$, we know that

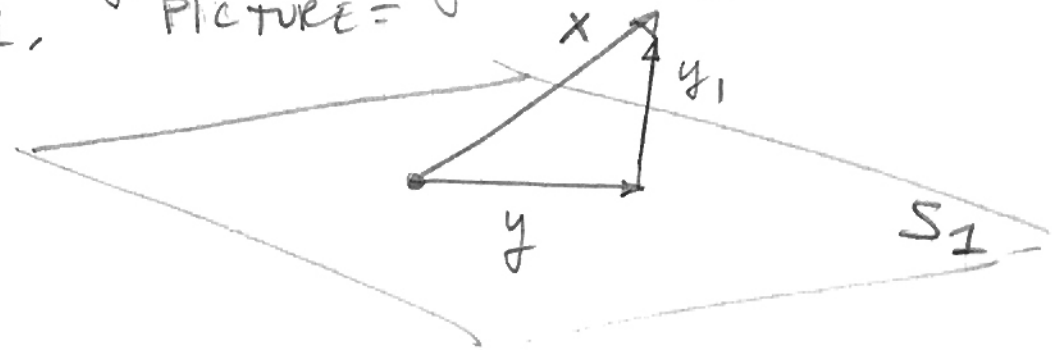
$$x_1 = \sum_{k=1}^{m} \beta_k \underline{e}_k ,$$

so that

$$x_1^T y_1 = \sum_{k=1}^{m} \sum_{j=m+1}^{n} \beta_k \alpha_j \underline{e}_k^T \underline{e}_j$$

$$= 0$$

That is y_1 is orthoganal to all vectors in S_1. PICTURE:



$\boxed{\text{PROBLEM 3}}$ DeRusso Problem 6-3

(a) $\mathcal{C}(A,B) = \begin{bmatrix} 1 & 7 \\ -1 & -4 \end{bmatrix} \Rightarrow$ controllable

(b) $\mathcal{O}(A,C) = \begin{bmatrix} 1 & 1 \\ -2 & -5 \end{bmatrix} \Rightarrow$ observable

(c) Order 2 Realization: Minimal. Order is based on # of states. Minimality is defined as completely controllable/observable representation.

PROBLEM 2 — DeRusso 6.2

(a) No. $\mathcal{C}(A,B) = \begin{bmatrix} 1 & -3+a \\ a & -2 \end{bmatrix}$

if $\det \mathcal{C} = 0$, not completely controllable

$$\begin{vmatrix} 1 & -3+a \\ a & -2 \end{vmatrix} = -2 + 3a - a^2 = 0$$

$$a^2 - 3a + 2 = 0$$

$$(a-1)(a-2) = 0$$

if $a = 1$ or $a = 2$, not completely controllable

(b) If $a = 1$ or 2, there is a pole-zero cancellation

PROBLEM 4 — DeRusso 6.9

(a)

$$M_{cc} = [A^2B \quad AB \quad B] \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}$$

where

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

is the characteristic equation

Characteristic Eqn = $\operatorname{Det} \begin{vmatrix} s+1 & 2 & 2 \\ 0 & s+1 & -1 \\ -1 & 0 & s+1 \end{vmatrix} = 0$

$$(s+1)^3 - 1(-2 - 2(s+1)) = 0$$

$$s^3 + 3s^2 + 3s + 1 + 2 + 2s + 1 = 0$$

$$s^3 + 3s^2 + 5s + 5 = 0$$

$$M_{cc} = \begin{bmatrix} 0 & -4 & 2 \\ 0 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 2 \\ 3 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} M_{cc}^{-1} A M_{cc} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -5 & -3 \end{bmatrix} \\ M_{cc}^{-1} B &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C M_{cc} &= \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \end{aligned}\right\} \text{controller canonical form.}$$

(b) $$y = \frac{2s^2 + 3s + 1}{s^2 + 3s^2 + 5s + 5}$$

(c) Since M_{cc} is full rank, system is completely controllable.

Also, note that there are no pole-zero cancellations (although these could indicate loss of observability.

$\boxed{\text{PROBLEM 5}}$ De Russo 6.12 (a)

Re-order the state vector as follows:

$$\text{(a)}\quad \begin{bmatrix}\dot{x}_3\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \\ \begin{bmatrix} -3 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \end{array} \begin{bmatrix}x_3\\ x_1\\ x_2\end{bmatrix} + \begin{bmatrix}2\\ 1\\ 0\end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix}x_3\\ x_1\\ x_2\end{bmatrix}$$

This is the Kalman Decomposition!
(eqn 6.8-17)

$\Rightarrow$ x_3 is $c\bar{o}$

x_1 is co

x_2 is $\bar{c}o$

(b) Minimal System is:

$$\dot{z} = z + u$$
$$y = z$$

$$\Rightarrow \boxed{y = \frac{1}{s-1} u}$$

PROBLEM 6 DeRusso 6.20

Controllability

$$\mathcal{C}(A,B) = \begin{bmatrix} b_1 & b_1+b_2 \\ b_2 & b_2 \end{bmatrix}$$

$$\det \mathcal{C} = b_1 b_2 - b_1 b_2 - b_2^2 = b_2^2$$

$b_2 \neq 0 \Rightarrow$ Completely Controllable

Observability

$$\mathcal{O}(A,C) = \begin{bmatrix} c_1 & c_2 \\ c_1 & c_1+c_2 \end{bmatrix}$$

$$\det \mathcal{O} = c_1^2 + c_1 c_2 - c_1 c_2 = c_1^2$$

$c_1 \neq 0 \Rightarrow$ Completely observable

PROBLEM 7 DeRusso 6.21

(a) $\mathcal{C}(A,B) = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \Rightarrow$ Completely Controllable

(b) $u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -K_1 x_1 - K_2 x_2$

$$\dot{\underline{x}} = \begin{bmatrix} -4-K_1 & -4-K_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det[sI-A] = 0$: $(s+4+K_1)s + (4+K_2)$

$$s^2 + (4+K_1)s + (4+K_2) = 0$$

For poles at $-4 \pm 2j$: (OVER)

$$[(s+4)-2j][(s+4)+2j]=0$$

$$(s+4)^2-(2j)^2=0$$

$$s^2+8s+16+4=0$$

$$s^2+8s+20=0$$

$$\Rightarrow \quad 4+K_1=8 \qquad \boxed{K_1=4}$$
$$4+K_2=20 \qquad \boxed{K_2=16}$$

PROBLEM 8 DeRusso 6.26 (a),(b),(c)

(a) $A_1 = \mathcal{C}(A,B) = \begin{bmatrix} b_1 & b_1 & b_1 \\ b_2 & b_2 & b_2 \\ b_3 & b_3 & b_3 \end{bmatrix}$

No choice of b_1, b_2, b_3 makes this system controllable

$A_2 = \mathcal{C}(A,B) = \begin{bmatrix} b_1 & b_1 & b_1 \\ b_2 & b_2+b_3 & b_2+2b_3 \\ b_3 & b_3 & b_3 \end{bmatrix}$

since $\mathrm{Det}(\mathcal{C}) = \mathrm{Det}(\mathcal{C}^T)$

and $\mathcal{C}^T$ has it's first and third columns linearly dependend

$\det(\mathcal{C}) = 0 \quad \forall\ b_1, b_2, b_3$

$\Rightarrow$ No choice of b_1, b_2, b_3 makes this system controllable

$A_3 = \mathcal{C}(A,B) = \begin{bmatrix} b_1 & b_1+b_2 & b_1+2b_2+b_3 \\ b_2 & b_2+b_3 & b_2+2b_3 \\ b_3 & b_3 & b_3 \end{bmatrix}$

$$\operatorname{Det}(\mathcal{C}) = \begin{vmatrix} b_1 & b_2 & 2b_2+b_3 \\ b_2 & b_3 & 2b_3 \\ b_3 & 0 & 0 \end{vmatrix}$$

(subtracting first column from second and third does not change determinant)

$$= b_3\left(2b_2b_3 - b_3(2b_2+b_3)\right)$$
$$= b_3^{\,3}$$

$b_3 \neq 0$ is necessary and sufficient for controllability!

(b) All three matrices have $\lambda_1 = \lambda_2 = \lambda_3 = 1$

But $(\lambda I - A_1) = [\emptyset] \Rightarrow$ Nullity $= 3$

And $(\lambda I - A_2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Nullity $= 2$

While $(\lambda I - A_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Nullity $= 1$

(Nullity = dimension − rank)

Since Nullity of $A_1 = 3$, any vector can be an eigenvector ∴ the I.C. response is thus independent of the I.C. ∴

$$\underline{x} = \underline{x}_0 e^t$$

For A_2 & A_3, the Jordan Blocks have Nullity < the dimension of the Block, so solutions to the D.E.s are more complicated. For A_2,

$\dot{x}_1 = x_1 \qquad \Rightarrow \qquad x_1 = x_{10} e^t$

$\dot{x}_2 = x_2 + x_3$

$\dot{x}_3 = x_3 \qquad \Rightarrow \qquad x_3 = x_{30} e^t$

$\dot{x}_2 = x_2 + x_{30} e^t$

$$x_2 = x_{20} e^t + x_{30} t e^t$$

$$\underline{x} = \underline{x}(0) e^t + \begin{bmatrix} 0 \\ x_3(0) \\ 0 \end{bmatrix} t e^t$$

Similarly, for A_3 we have more complex initial condition response, because there are insufficient distinct eigenvalues

(c) If Nullity $= 1$, the interdependence of the state responses [e.g. $x_1(t)$, $x_2(t)$, and $x_3(t)$] allows controllability to be maintained.

If Nullity > 1, their is an "ambiguity" between the states; it is not clear from the response what states are responsible.

$\therefore$ Controllability is <u>lost</u>

PROBLEM 9 De Russo 7.3

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

$$\boxed{\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky}$$

This is the observer; all that is left to do is specify K

$$A - KC = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 & 1 \\ 9 - k_2 & 0 \end{bmatrix}$$

eigenvalues: $\text{Det} \begin{vmatrix} s + k_1 & -1 \\ k_2 - 9 & s \end{vmatrix} = 0$

$$s^2 + k_1 s + k_2 - 9 = 0$$

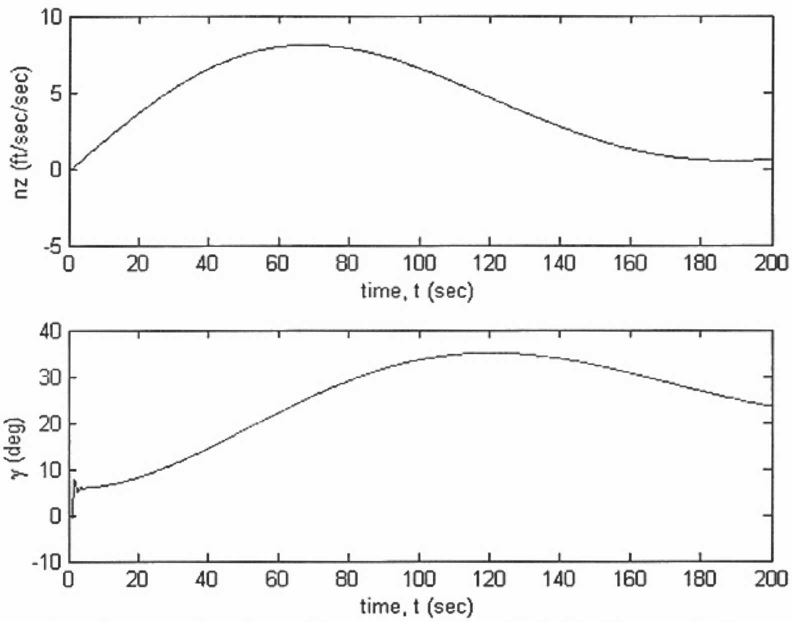
desired characteristic equation:

$$[(s+10) + 10j][(s+10) - 10j] = 0$$

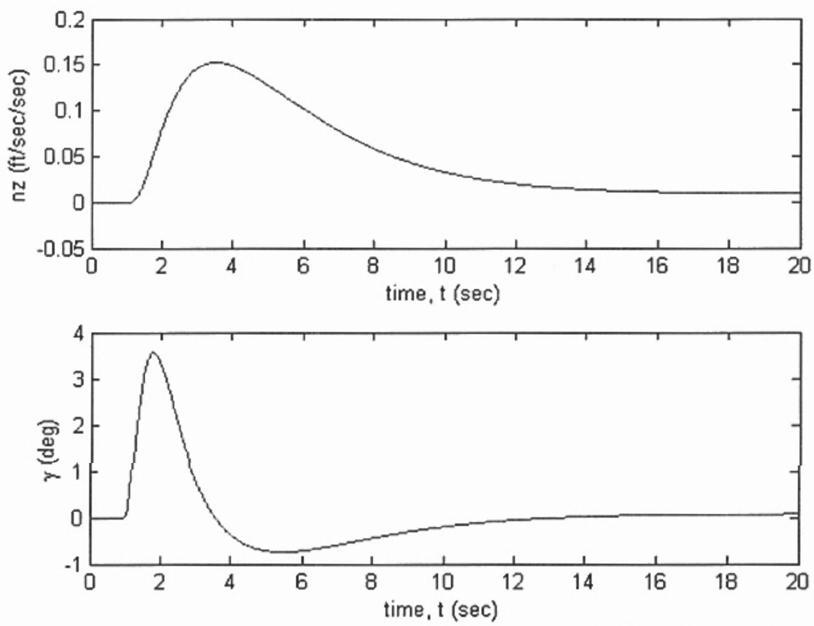
$$s^2 + 20s + 100 + 100 = 0$$

$$s^2 + 20s + 200 = 0$$

$$\Rightarrow \boxed{\begin{aligned} k_1 &= 20 \\ k_2 &= 209 \end{aligned}}$$



(a) The first plot shows that there is a very slow, lightly damped eigenvalue that dominates in the nz response. This is called the phugoid mode. Initially reacts in the 'short period', or fast eigenvalue, oscillating around a final value of ~ 6 degrees for about 2 seconds before settling down. However, after this there is an even stronger influence of the phugoid in the response.

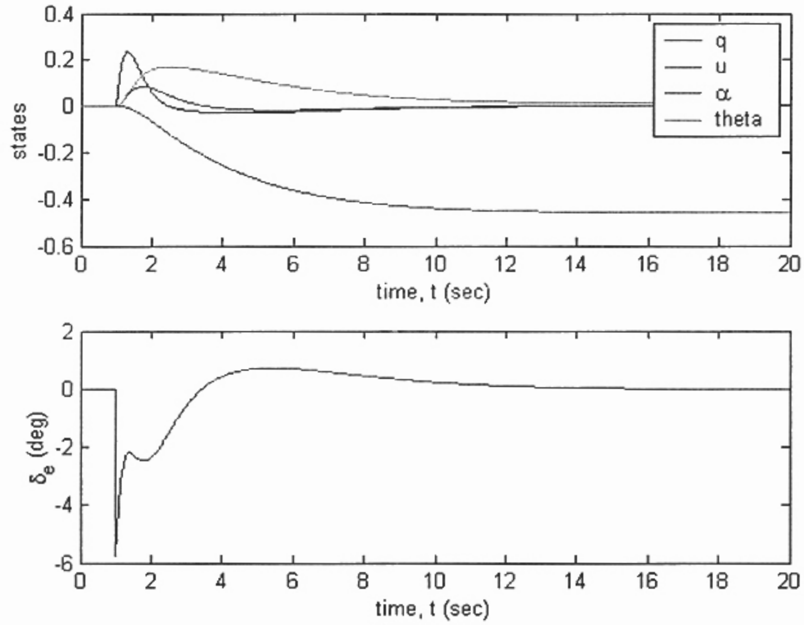


(b) The second plot shows the effect of feedback with the eigenvalues chosen at

[-3+2j, -3+2j, -0.5, -0.5].

Note that although the oscillations are gone, the command response is not good. In fact we have not really even specified our desires for command response, so this is understandable!

(b) continued: State and input responses:



K = [-0.2505 0.2092 0.4446 -0.4103]

Note that we seem to be commanding –u ! This is consistent with the idea that elevator input eventually drives the system to a new trim velocity.

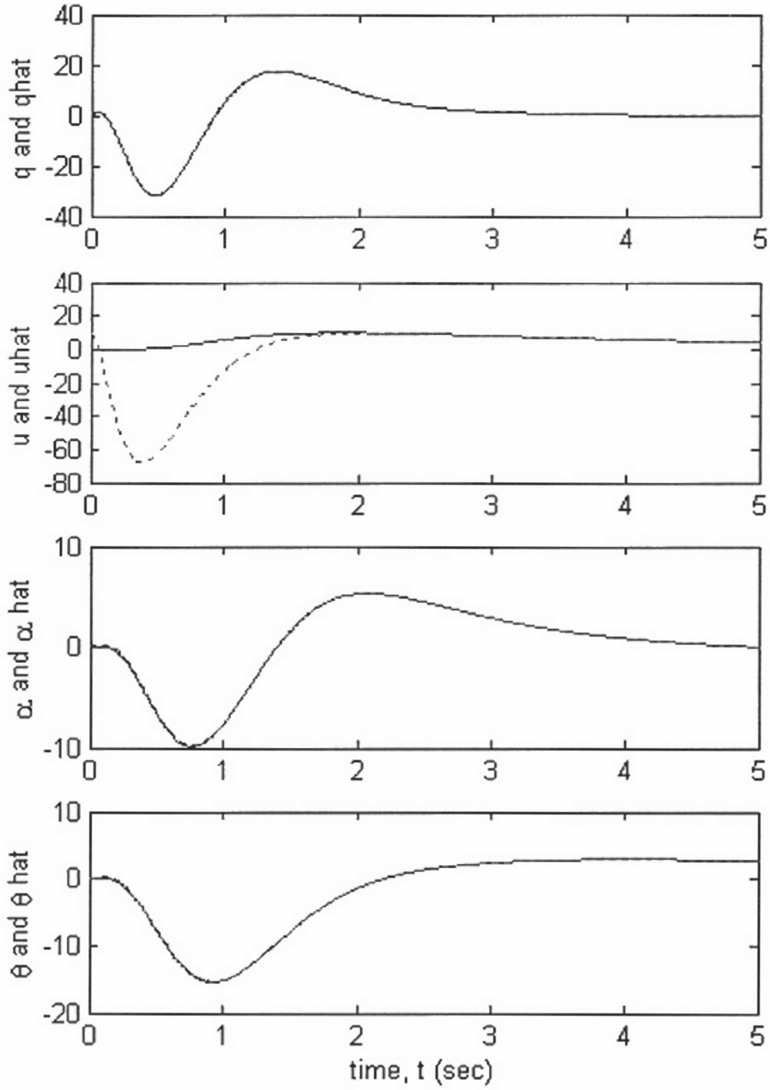
(c) Estimator Equation: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$

K matrix (see Matlab file):

$$\begin{bmatrix} 126.98 \\ 36979 \\ -76.625 \\ -58.939 \end{bmatrix}$$

(d)

See At, Bt, Ct, and Dt matrix equations in attached m-file to see how all the matrices combine into an overall system description. The time history is shown on the next page.



Errors in the state estimates generate large errors in that propagate through the transient response. All states estimates converge to the actual values quickly except for u, whose estimate takes somewhat longer to die out.

```
% use first or second control input; first or second measurement
nu=1;
ny=1;

% (a) System Matrices states are (q u alpha theta)
A = [-0.8   -0.0344   -12        0
       0    -0.014    -0.2904   -0.562
       1    -0.0057   -1.5       0
       1     0          0        0 ];

 B = [-19       -2.5
      -0.0115   -0.0087
      -0.16     -0.6
       0         0     ];

 C = [ 0    0    -1       1
       0    0     0.733   0 ];

 D = [ 0         0
       0.0768   .1134 ];

 sys=ss(A,B(:,nu),C,D(:,nu));
 T = [0:.01:200]';
 U = [zeros(1,100),-.1*ones(1,19901)]';

y = lsim(sys,U,T,zeros(4,1));

figure(1)
subplot(211)
plot(T,y(:,1))
xlabel('time, t (sec)')
ylabel('nz (ft/sec/sec)')
subplot(212)
plot(T,180/pi*y(:,2))
xlabel('time, t (sec)')
ylabel('\gamma (deg)')

% (b) Pick input and output; convert to control canonical form
%    (Full state feedback does not need Cc, but we'll make it anyway, Ct is
 'temporary')
B=B(:,nu);
Ct=C(ny,:);
D=0;
[num,den]=ss2tf(A,B,Ct,D);

Ac=[ 0 1 0 0
     0 0 1 0
     0 0 0 1
    -den(5) -den(4) -den(3) -den(2)];
```

```
Bc=[0;0;0;1];
Cc=[num(5) num(4) num(3) num(2)];
Dc=0;

% (b) cont'd Determine Gains
%    let K = [k4 k3 k2 k1]
%    u = -Kx ==>  Acl = A-BK ==> A - [ 0;0;0;k4 k3 k2 k1]
%    so characteristic equation is s^4 + (den(2)+k1)*s^3 + (den(3)+k2)*s^2 ↙
+ (den(4)+k3)*s + den(5)+k4
%
% Choose eigenvalue locations and build up desired characteristic equation

i=sqrt(-1);
des_eigs = [-3+2*i, -3-2*i, -0.5, -0.5];
des_chic = poly(des_eigs);
Kc = des_chic(2:5)-den(2:5);
Kc = Kc(4:-1:1);
eig(Ac-Bc*Kc);  % Check answers

% (b) cont'd Need to translate gains back to original states!
%    xnew = T*xold
%    so K*xnew = K*T*xold;   Need to find T!!!!  where T*A*Tinv = Ac

a1 = den(2); a2=den(3); a3=den(4);

Mcc=[1    0    0    0
     a1    1    0    0
     a2   a1   1    0
     a3   a2   a1   1  ];
Mcc=[A^3*B A^2*B A*B B]*Mcc;
Ac - inv(Mcc)*A*Mcc;    % Check that the transformation works
                        % This implies that T=inv(Mcc)

K = Kc*inv(Mcc);
eig(A-B*K)

sys=ss(A-B*K,B,C,D);
T = [0:.01:20]';
U = [zeros(1,100),-.1*ones(1,1901)]';
[y,t,x] = lsim(sys,U,T,zeros(4,1));

figure(2)
subplot(211)
plot(T,y(:,1))
xlabel('time, t (sec)')
ylabel('nz (ft/sec/sec)')
subplot(212)
plot(T,180/pi*y(:,2))
```

```
xlabel('time, t (sec)')
ylabel('\gamma (deg)')

u = (-K*x')'+ U;
figure(3)
subplot(211)
plot(t,x)
xlabel('time, t (sec)')
ylabel('states')
legend('q','u','\alpha','theta')
subplot(212)
plot(t,180/pi*u)
xlabel('time, t (sec)')
ylabel('\delta_e (deg)')


% =====================================================
%
% (c) Now, can we estimate x based on the measurements?
%     xhat_dot = A*xhat + B*u + Ke*(y-C*xhat)

%  Convert to observer form immediately
C = C(ny,:);
Moc = [ a3 a2 a1 1
        a2 a1 1  0
        a1  1 0  0
         1  0 0  0 ];

Moc = Moc*[C; C*A; C*A^2; C*A^3];
Ao = Moc*A*inv(Moc);
Co = [0 0 0 1];

% Choose eigenvalues of observer and find observer gain matrix
des_oeigs = [-5 -5 -5 -5];
des_ochic = poly(des_oeigs);
Ko = des_ochic(2:5)-den(2:5);
Ko = Ko(4:-1:1)';
eig(Ao-Ko*Co);    % Check answer

% Convert back to original state space form
%
Ke = inv(Moc)*Ko;
eig(A-Ke*C);     % Check

% (d) FINALLY:: Build up entire system and simulate

At = [   A      -B*K
        Ke*C   A-B*K-Ke*C  ];
Bt = [   B ;    B ];
```

```
Ct = [   C 0 0 0 0];
Dt = 0;

eig(At)

sys=ss(At,Bt,Ct,Dt);
[yt,t,x] = lsim(sys,U,T,[0 0 0 0 0.5 10 0.1 0.1]');

figure(4)
subplot(411)
plot(t,x(:,1),t, x(:,5),':')
ylabel('q and qhat')
subplot(412)
plot(t,x(:,2),t, x(:,6),':')
ylabel('u and uhat')
subplot(413)
plot(t,x(:,3),t, x(:,7),':')
ylabel('\alpha and \alpha hat')
subplot(414)
plot(t,x(:,4),t, x(:,8),':')
ylabel('\theta and \theta hat')
xlabel('time, t (sec)')
```

Problem 8 DeRusso et al. 6.26 (a), (c)

$\boxed{\text{PBH Solution}}$

(a)
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B} \quad \underbrace{\left[\begin{array}{ccc|c} s-1 & 0 & 0 & b_1 \\ 0 & s-1 & 0 & b_2 \\ 0 & 0 & s-1 & b_3 \end{array}\right]}_{[sI-A \,\vdots\, B]}$$

$s=1 \Rightarrow \text{rank} < 3 \quad \forall\, b_1, b_2, b_3$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B} \quad \underbrace{\left[\begin{array}{ccc|c} s-1 & -1 & 0 & b_1 \\ 0 & s-1 & 0 & b_2 \\ 0 & 0 & s-1 & b_3 \end{array}\right]}_{[sI-A \,\vdots\, B]}$$

$s=1 \Rightarrow \text{rank} < 3 \quad \forall\, b_1, b_2, b_3$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B} \quad \underbrace{\left[\begin{array}{ccc|c} s-1 & -1 & 0 & b_1 \\ 0 & s-1 & -1 & b_2 \\ 0 & 0 & s-1 & b_3 \end{array}\right]}_{[sI-A \,\vdots\, B]}$$

$s=1 \Rightarrow \text{rank} < 3 \quad \text{if } b_3 = \emptyset$

(c) For a repeated eigenvalue must have 1 chain of independent eigenvectors for single control or must have one control for each lin. indep eigenvector chain.