Homework #6 Solutions

Problem 1

$$G(s) = \frac{s+2}{s^2-1}$$

controllable form =

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

gen. eig problem =

$$\underbrace{\begin{bmatrix} s_0 & -1 & 0 \\ -1 & s_0 & -1 \\ 2 & 1 & 0 \end{bmatrix}}_{V}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det(V) = 0 \Rightarrow$ (expand around last column)

$$-1(s_0+2) = 0$$

$$s_0 = -2 \checkmark$$

Direction:

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & -1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \omega \end{bmatrix} = 0$$

$$\left.\begin{array}{ll} \theta_2 = -2\theta_1 & \Rightarrow \theta = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ w = -\theta_1 - 2\theta_2 & \Rightarrow w = 3 \end{array}\right\}$$

So let $u = 3e^{-2t}$

$$x_1(0) = 1, \quad x_2(0) = -2$$

(over)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned}\right\} \quad \ddot{x}_2 = x_2 + \dot{u}$$ ②

particular soln for $u = 3e^{-2t}$

$$x_{2_p} = Ce^{-2t}$$

$$+4Ce^{-2t} = Ce^{-2t} - 6e^{-2t}$$

$$3C = -6 \qquad \Rightarrow \quad C = -2$$

homogeneous soln

$$x_{2_H} = C_1 e^{t} + C_2 e^{-t}$$

Combine x_{2p} & x_{2H} and solve for C_1 & C_2 using the I.C.'s from the gen. eig problem:

$$x_2(t) = C_1 e^{t} + C_2 e^{-t} - 2e^{-2t}$$

$$x_2(0) = C_1 + C_2 - 2 = -2$$

$$\dot{x}_2(0) = C_1 - C_2 + 4 = x_1(0) + u(0)$$

$$= 1 + 3$$

$$\Rightarrow \quad \begin{aligned} C_1 + C_2 &= 0 \\ C_1 - C_2 &= 0 \end{aligned} \longrightarrow C_1 = C_2 = 0$$

$$\therefore \quad x_2(t) = -2e^{-2t}$$

$$x_1(t) = \int x_2(t)\,dt = e^{-2t} + C_3$$

$$x_1(0) = 1 + C_3 = 1 \Rightarrow C_3 = 0$$

FINALLY, we Have: $y = 2x_1 + x_2$

$$= 2e^{-2t} - 2e^{-2t} = 0$$

$\boxed{\text{PROBLEM 2}}$ (a)

$$\det\begin{bmatrix} s_0-2 & 2 & -3 & -4 & 1 \\ -1 & s_0-1 & -1 & -2 & 1 \\ -1 & -3 & s_0+1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = 0$$

<u>expand about last row:</u>

$$\det\begin{bmatrix} s_0-2 & -3 & -4 & 1 \\ -1 & -1 & -2 & 1 \\ -1 & s_0+1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 0$$

<u>expand again about last row:</u>

$$\det\begin{bmatrix} -3 & -4 & 1 \\ -1 & -2 & 1 \\ s_0+1 & -1 & 1 \end{bmatrix} = 0$$

$$6 - 4(s_0+1) + 1 + 2(s_0+1) - 3 - 4 = 0$$

$$\Rightarrow \quad s_0 = -1$$

<u>Direction:</u>

$$\begin{bmatrix} -3 & 2 & -3 & -4 & 1 \\ -1 & -2 & -1 & -2 & 1 \\ -1 & -3 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{w} \end{bmatrix} = 0$$

$$\left.\begin{array}{l} \theta_2 = 0 \\ \theta_1 = 0 \\ w_1 = w_2 \\ -\theta_3 - w_1 = 0 \end{array}\right\} \quad \underline{\theta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underline{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

only the 3rd state participates in this NMP zero. If $u(t) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{t}$ (NOTE: diverging!) and $\underline{X}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $y(t) \equiv 0 \;\; \forall t$.

(b) Using Matlab "ss2tf", we have

$$G(s) = \frac{1}{(s-1)(s+2)(s-3)} \begin{bmatrix} 4s^2 - s - 7 & -(s+2)(s-1) \\ 2s^2 + 3s - 1 & -(s+2)(s-1) \end{bmatrix}$$

The SISO transfer function zeroes are <u>not</u> <u>necessarily</u> at the transmission zeros. Not however that $G(s)$ is rank deficient

(c) First, perform an eigenvector decomposition of A, using Matlab "eig"

$$A = V^{-1} \Lambda V = W \Lambda V$$

for $\lambda_1 = 1$ $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

To be unobservable we need

$$C \underline{v}_1 = 0 \; ; \text{ choose } C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Using Matlab's "Eig" to find the Generalized eigenvectors, we have

$$S_0 = 1 \quad \text{(as expected)}$$

$$\underline{\theta} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{w} = \underline{0}$$

The generalized eigenvector <u>matches</u> the eigenvector for the cancelled zero. Any <u>I.C.</u> in eigenvector direction gives $y(t) = 0$.

(d) to regain observability, we need only move the unobservable pole since the zero will not change. Let

$$K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ for simplicity.

then the C.L. system is

$$\dot{\underline{x}} = (A - BK)\underline{x} + B\underline{r}$$

$$\underline{y} = C\underline{x}$$

<u>Observability is given by</u>

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$ which one can check, in matlab, is <u>full rank</u>

(e) If $u = -K\underline{y}$ then the mode associated with the zero will not be effected (it does not appear in the output)

$\therefore$ output feedback cannot recover observability

ALTHOUGH state feedback <u>can</u> recover obserability as shown in (c)

Problem 3 Bélanger # 7.5

(a) $\mathcal{C}(A, B) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{rank} = 1 \Rightarrow$ uncontrollable

$$\det \begin{bmatrix} \lambda & -1 \\ 0 & \lambda+1 \end{bmatrix} = 0 \Rightarrow \lambda(\lambda+1) = 0$$

poles @ $\lambda_1 = 0$
$\lambda_2 = -1$

eigenvectors: $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$v_2 = 0 \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$v_1 = -v_2 \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$V = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{matrix} w_1^T = [1 \quad 1] \\ w_2^T = [0 \quad -1] \end{matrix}$$

uncontrollable mode has

$$\underline{w}_i^T B = 0$$

Since $w_1^T B = 0$ $\lambda_1 = 0$ is uncontrollable

(b) $$\dot{x} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} [k_1 \quad k_2] \right\} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{ex}$$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ -k_1 & -1-k_2 \end{bmatrix} \underline{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{ex}$$

(c) $\mathcal{C}(A_{c.l.}, B) = \begin{bmatrix} -1 & -k_1+k_2+1 \\ 1 & k_1-k_2-1 \end{bmatrix}$

note that columns are <u>still</u> linearly <u>dependent</u>, no matter what k_1 & k_2 are.

<u>Controllability has not been recovered</u>

$$\begin{bmatrix} \lambda-k_1 & -1-k_2 \\ k_1 & \lambda+1+k_2 \end{bmatrix} = 0$$

$$(\lambda-k_1)(\lambda+1+k_2) + k_1(1+k_2) = 0$$

$$\lambda^2 - \lambda(1+k_2-k_1) - \cancel{k_1(1+k_2)} + \cancel{k_1(1+k_2)} = 0$$

$$\lambda(\lambda - (1+k_2-k_1)) = 0$$

eigenvalue at $\lambda=0$ does not change.

<u>left eigenvector for $\lambda_1=0$</u>

$$[w_1 \quad w_2]^T \begin{bmatrix} -k_1 & -1-k_2 \\ k_1 & 1+k_2 \end{bmatrix} = 0$$

$$-k_1 w_1 + k_1 w_2 = 0 \Rightarrow w_1 = w_2$$

$\underline{w}_1^T = [1 \quad 1]$ (as before) $\underline{w}_1^T B = 0$

$\lambda_1 = 0$ <u>STILL UNCONTROLLABLE</u>

(d) $$\dot{x} = \begin{bmatrix} k_1 & 1+k_2 \\ -k_1 & -1-k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{ex}$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find T.F. $y/u_{ex} =$

$$s x_1 = k_1 x_1 + \ell_1 x_2 - u_{ex} \qquad (\text{where } \ell_1 = k_2 + 1)$$

$$\boxed{x_1 = \left(\frac{1}{s-k_1}\right)(\ell_1 x_2 - u_{ex})}$$

$$s x_2 = -k\left(\frac{1}{s-k_1}\right)(\ell_1 x_1 - u_{ex}) - \ell_1 x_2 + u_{ex}$$

$$0 = -(s-k_1) s x_2 - k_1 \ell_1 x_2 + k_1 u_{ex}$$
$$-\ell_1 (s-k_1) x_2 + (s-k_1) u_{ex}$$
$$= \left[-s^2 + (k_1 - \ell_1) s + (-k_1\ell_1 + k_1\ell_1)\right] x_2$$
$$+ \left[s - k_1 + k_1\right] u_{ex}$$

$$\Rightarrow \boxed{x_2(s) = \frac{s}{s^2 - (k_1 - \ell_1) s} u_{ex}}$$

and $$x_1(s) = \left(\frac{1}{s-k_1}\right)\left(\frac{\ell_1}{s-(k_1-\ell_1)} - 1\right) u_{ex}$$
$$= \left(\frac{1}{s-k_1}\right)\left(\frac{\ell_1 - (s - k_1 + \ell_1)}{s-(k_1-\ell_1)}\right) u_{ex}$$

$$\boxed{x_1(s) = \frac{1}{s-(k_1-\ell_1)} u_{ex}}$$

$$\boxed{y/u_{ex} = \left(\frac{c_1 s + c_2}{s-(k_1-\ell_1)}\right)}$$

The pole $\lambda_1 = 0$ does not appear in either x_1 or x_2, so it cannot appear in $y(s)$ [illegible]

PROBLEM 4

Bélanger Problem 7.18 w/ $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C = [1 \;\; -1]$

(a) $u = -Ky = -[k \;\; -k]\,\underline{x} = -K\underline{x}$

$$\dot{x} = (A - BK)x = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k \;\; -k] \right\} x$$

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -k & k-1 \end{bmatrix} \underline{x}$$

(b) $$\int_0^\infty y^2(t)\,dt = \int_0^\infty \underline{x}^T C^T C \underline{x}\,dt = \int_0^\infty \underline{x}^T \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \;\; -1]\,\underline{x}$$

$$= \int_0^\infty \underline{x}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{x}\,dt \triangleq \int_0^\infty \underline{x}^T S \underline{x}\,dt$$

Using Eqn 7.24:

$$J = \int_0^\infty y^2(t)\,dt = x_0^T P x_0 \quad \text{where } P \text{ solves}$$

$$A^T P + PA = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

expanding:

$$\begin{bmatrix} 0 & -k \\ 1 & k-1 \end{bmatrix} \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k & k-1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -kp_3 & -kp_2 \\ p_1 + (k-1)p_3 & p_3 + (k-1)p_2 \end{bmatrix} + \begin{bmatrix} -kp_3 & p_1 + (k-1)p_3 \\ -kp_2 & p_3 + (k-1)p_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

(1,1) eqn: $-2kp_3 = -1$

(2,2) eqn: $2p_3 + 2(k-1)p_2 = -1$

$\Rightarrow 2k(k-1)p_2 = -k-1$

$$p_2 = \frac{-(k+1)}{2k(k-1)}$$

(1,2) eqn $-kp_2 + p_1 + (k-1)p_3 = 1$

$$\left.\begin{aligned} P_3 &= \frac{1}{2K} \\ P_2 &= \frac{-(K+1)}{2K(K-1)} \end{aligned}\right\} \Rightarrow \text{substitute into } (1,2) = \frac{+(K+1)}{2(K-1)} + P_1 + \frac{(K-1)}{2K} = 1$$

$$P_1 = 1 - \frac{(K+1)}{2(K-1)} - \frac{(K-1)}{2K}$$

$$= \frac{2K(K-1) - K(K+1) - (K-1)^2}{2K(K-1)}$$

$$P_1 = \frac{\cancel{2K^2} - \cancel{2K} - \cancel{K^2} - K - \cancel{K^2} + \cancel{2K} - 1}{2K(K-1)}$$

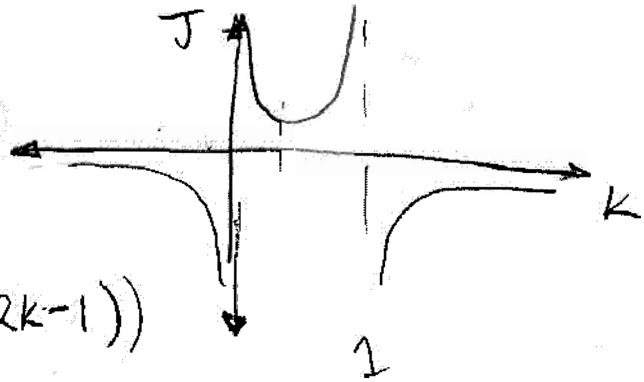
$$= \frac{-K-1}{2K(K-1)} = P_2$$

$$J = x_0^T P x_0 = \begin{bmatrix} x_{10} & x_{20} \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3 & P_1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$= \begin{bmatrix} x_{10} & x_{20} \end{bmatrix} \begin{bmatrix} P_1 x_{10} + P_3 x_{20} \\ P_1 x_{20} + P_3 x_{10} \end{bmatrix}$$

$$= P_1 (x_{10}^2 + x_{20}^2) + 2P_3 (x_{10} x_{20})$$

(c) $$J(K) = \frac{-(K+1)}{2K(K-1)} X_{10}^2$$



$$\frac{\partial J}{\partial K} = -\frac{X_{10}^2}{2} \left((K+1)\left(\frac{1}{K^2(K-1)^2}(2K-1)\right) - \frac{1}{K(K-1)} \right) = 0$$

$$\Rightarrow (K+1)(2K-1) - K(K-1) = 0$$

$$2K^2 + K - 1 - K^2 + K = 0$$

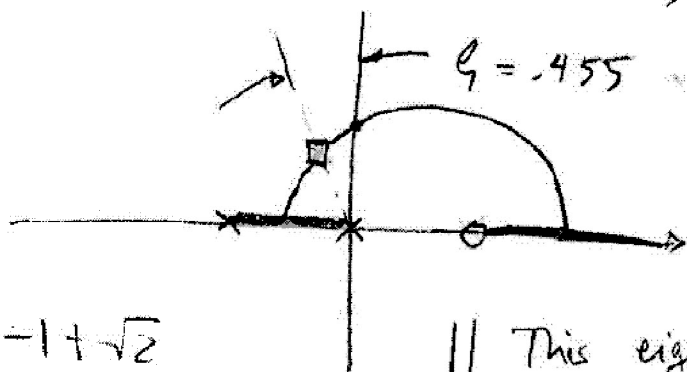
$$K^2 + 2K - 1 = 0$$

$$K = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$K > 0$ needed for pos def. P

$$\boxed{\therefore\ K = -1 + \sqrt{2}}$$

minimizes J, for <u>any</u> I.C.

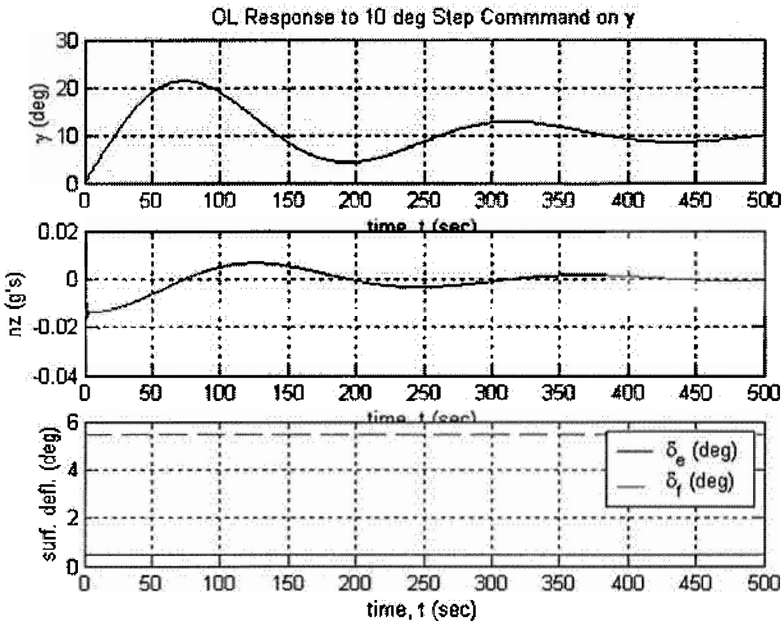


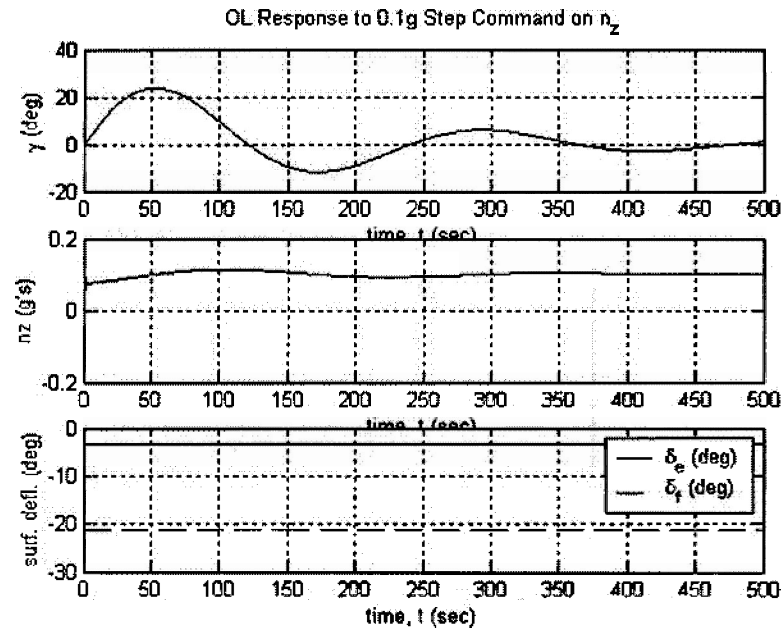
for $K = -1 + \sqrt{2}$

$eig(a - bkc) = .29 \pm .57j$

This eigenvalue represents the optimum trade-off between speed of response and damping

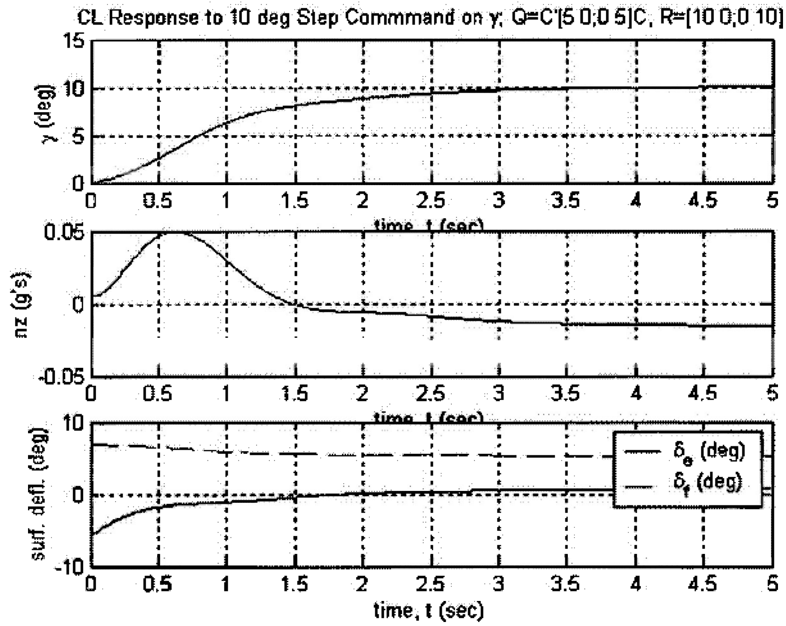
Problem 5(a)

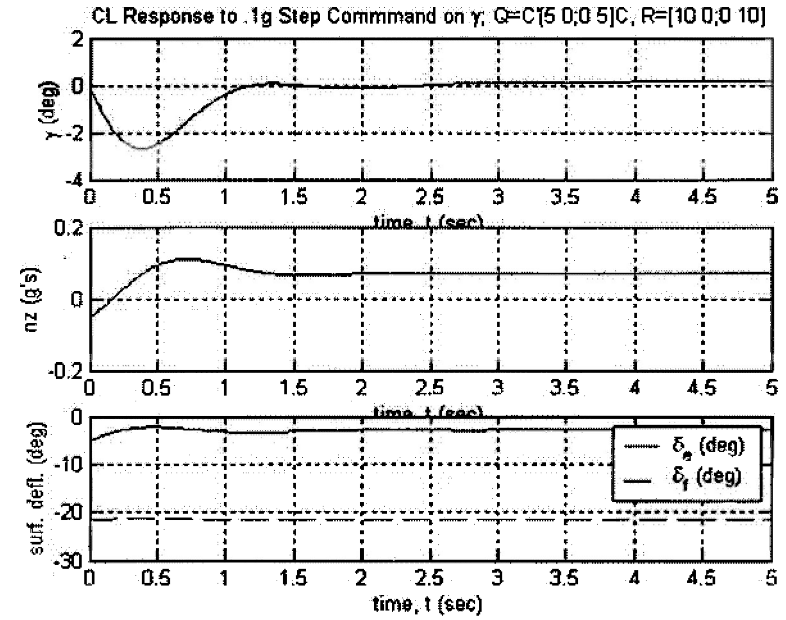




Note that (i) the surface deflections are constant, and (ii) after a very long transient the outputs go to their commanded values. The open loop system is stable, but there is a very slow, underdamped eigenvalue.

Problem 5(b)





Note here that the control inputs go through very slight transient deviations from their trim values to damp out the underdamped modes, and achieve nice command-following properties. Note also that nz does not seem to achieve the desired steady-state value. This is due to a transmission zero near the origin! Over a very long time scale, nz does in fact converge, as it must. To eliminate this problem, an integrator in the feedback loop, or some other form of low-frequency *compensation*, is necessary. We've done all we can with full-state feedback!

```
nz_com  = .1;              % input nz magnitude
gam_com = 10*pi/180;       % input gamma mag.

% System Matrices states are (q u alpha theta)
A = [-0.8  -0.0344  -12      0
      0   -0.014   -0.2904  -0.562
      1   -0.0057  -1.5      0
      1    0        0        0 ];

B = [-19      -2.5
     -0.0115  -0.0087
     -0.16    -0.6
      0        0   ];

C = [ 0  0  -1     1
      0  0   0.733 0 ];

D = [ 0       0
      0.0768  .1134 ];

% (a) Compute Feedforward matrices
M = [ A  B
      C  D ];
M =  inv(M);
Mx = M(1:4,5:6);
Mu = M(5:6,5:6);

% (a) Create O.L. system, simulate with nz and
%     gamma steps
sys=ss(A,B*Mu,C,D*Mu);
T = [0:.01:500]';

U = [gam_com*ones(length(T),1)
0*nz_com*ones(length(T),1)];
y = lsim(sys,U,T,zeros(4,1));
figure(1)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('OL Response to 10 deg Step Commmand on
\gamma')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g"s)'), grid
dedf=(Mu*U')'*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'),
grid
legend('\delta_e (deg)','\delta_f (deg)')
% ---

U = [0*gam_com*ones(length(T),1)
nz_com*ones(length(T),1)];
y = lsim(sys,U,T,zeros(4,1));
figure(2)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('OL Response to 0.1g Step Command on n_z')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g"s)'), grid
dedf=(Mu*U')'*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'),
grid
legend('\delta_e (deg)','\delta_f (deg)')

% (b) Compute LQR gains and create C.L.
system, then re-simulate
%
Qy= diag([5 5]);
Q = C'*Qy*C;
R = diag([10 10]);

K = lqr(A,B,Q,R);

sys = ss(A-B*K, B*(Mu+K*Mx), C-D*K,
D*(Mu+K*Mx));
T = [0:.01:5]';

U = [gam_com*ones(length(T),1)
0*nz_com*ones(length(T),1)];
[y,t,x] = lsim(sys,U,T,zeros(4,1));
figure(3)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'),
grid
title('CL Response to 10 deg Step Commmand
on \gamma; Q=C"[5 0;0 5]C, R=[10 0;0 10]')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g"s)'), grid
dedf=( ((Mu+K*Mx)*U')'-(K*x')' )*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'),
grid
legend('\delta_e (deg)','\delta_f (deg)')
% ---

U = [0*gam_com*ones(length(T),1)
nz_com*ones(length(T),1)];
[y,t,x] = lsim(sys,U,T,zeros(4,1));
figure(4)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'),
grid
title('CL Response to .1g Step Commmand on
\gamma; Q=C"[5 0;0 5]C, R=[10 0;0 10]')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g"s)'), grid
dedf=( ((Mu+K*Mx)*U')'-(K*x')' )*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'),
grid
legend('\delta_e (deg)','\delta_f (deg)')
```