HOMEWORK 7 SOLUTIONS

PROBLEM 1 Bélanger 7.20

$K = R^{-1}B^TP$, where P is the soln of

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = \rho$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1/\rho \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -p_3 & -p_2 \\ p_1 & p_3 \end{bmatrix} + \begin{bmatrix} -p_3 & p_1 \\ -p_2 & p_3 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_2 \end{bmatrix} 1/\rho \begin{bmatrix} p_3 & p_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2p_3 & p_1 - p_2 \\ p_1 - p_2 & 2p_3 \end{bmatrix} - \frac{1}{\rho}\begin{bmatrix} p_3^2 & p_3p_2 \\ p_2p_3 & p_2^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(1,1) equation: $-2p_3 - p_3^2/\rho + 1 = 0$

$$p_3^2 + 2\rho p_3 - \rho = 0$$

$$p_3 = \frac{-2\rho \pm \sqrt{4\rho^2 + 4\rho}}{2}$$

$$= -\rho + \sqrt{\rho^2 + \rho}$$

(2,2) equation $2p_3 - 1/\rho \, p_2^2 = 0$

$$p_2 = \sqrt{2\rho p_3} = \sqrt{-2\rho^2 + 2\rho\sqrt{\rho^2+\rho}}$$

$$K = \frac{1}{\rho}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_3 \\ p_3 & p_2 \end{bmatrix}$$

$$= \frac{1}{\rho}\begin{bmatrix} p_3 & p_2 \end{bmatrix}$$

$$\boxed{K = \begin{bmatrix} -1+\sqrt{1+1/\rho} & \sqrt{-2+2\sqrt{1+1/\rho}} \end{bmatrix}}$$

$$J_u = \int_0^\infty u^2\, dt = \int_0^\infty x^T K^T K x\, dt$$

$= x_0^T P_u x_0$, where P_u is the solution of

$$A_{cl}^T P_u + P_u A_{cl} = K^T K \quad , \quad A_{cl} = A - BK$$

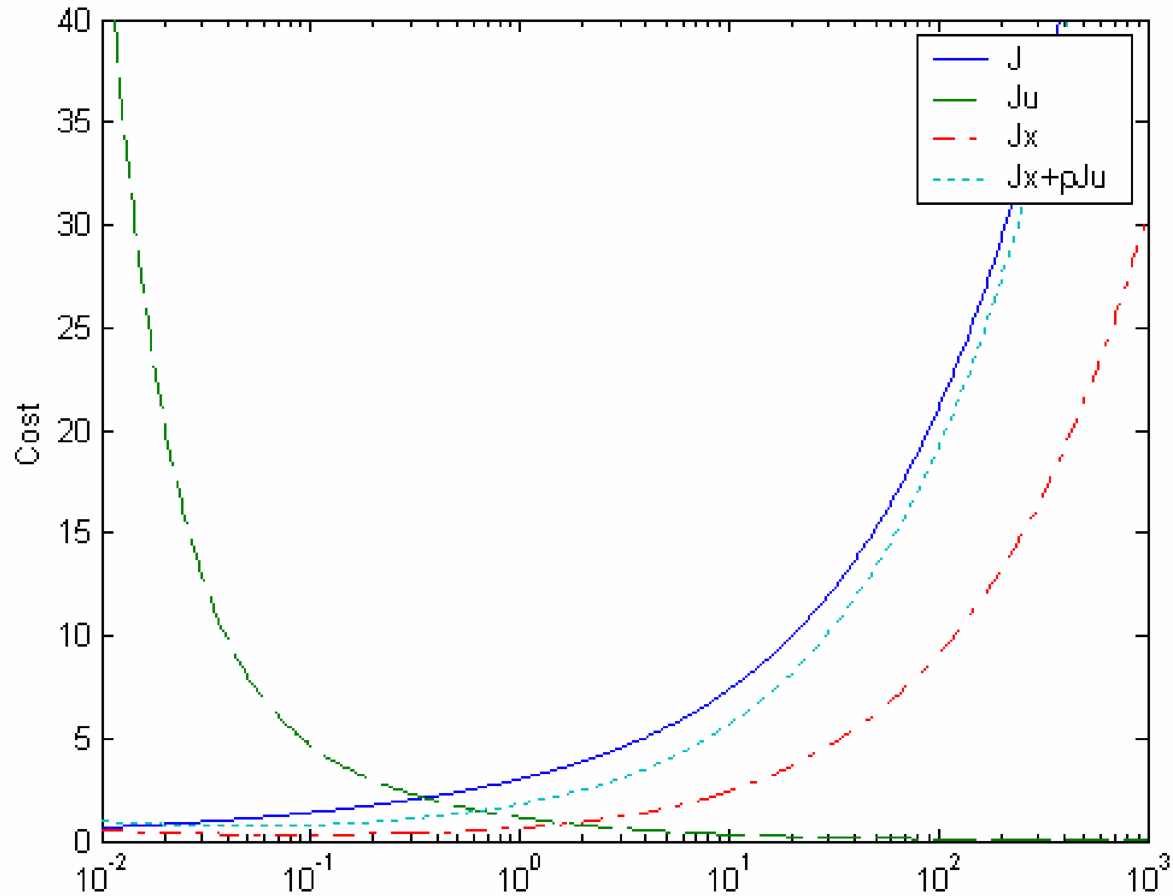
$$J_x = \int_0^\infty x^2\, dt = \int_0^\infty x^T Q x\, dt$$

$= x_0^T P_x x_0$, where P_x is the soln of

$$A_{cl}^T P_x + P_x A_{cl} = Q$$

See plots of J_u, J_x, and $J_x + \rho J_u$ attached.

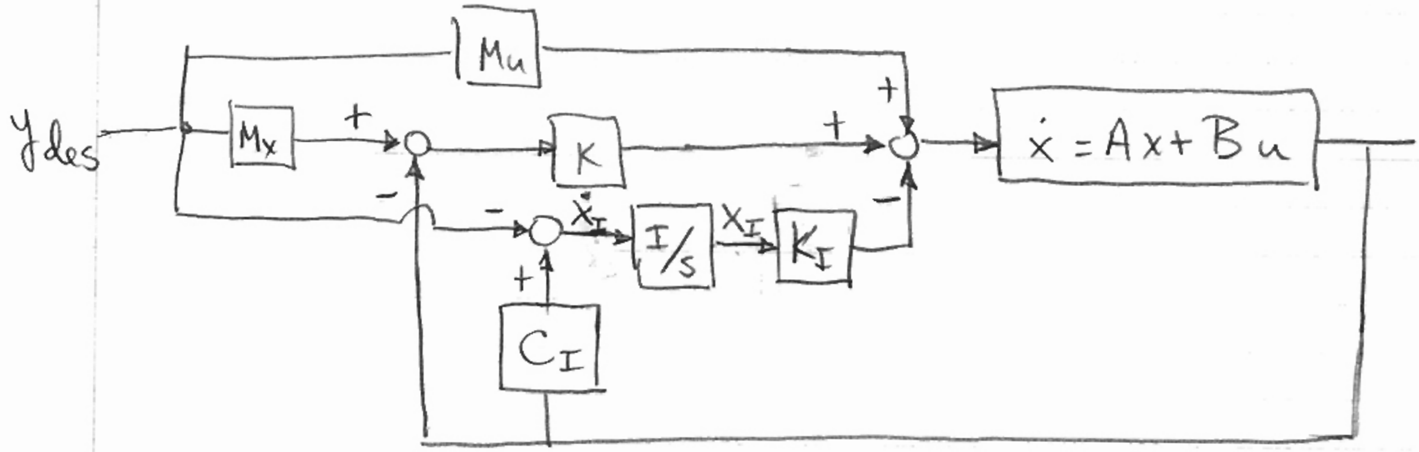
Problem 1, Numerical Part




```
A=[0 1;-1 0];
B=[0;1];
Q=[1 0;0 0];
x0=[1;1];

rho=logspace(-2,3,500);

for i=1:length(rho)
   R=rho(i);
   [K,P]  = lqr(A,B,Q,R);
   Pu = lyap(A-B*K, K'*K);
   Px = lyap(A-B*K, Q);

   J(i) = x0'*P*x0;
   Ju(i) = x0'*Pu*x0;
   Jx(i) = x0'*Px*x0;
   JJ(i) = Jx(i) + R*Ju(i);
end

semilogx(rho,J,rho,Ju,rho,Jx,rho,JJ)
legend('J','Ju','Jx','Jx+\rhoJu')
axis([.01,1000,0 40])
```

Note that the cyan plot and the blue plot should be on top of each other – J is computed using the Ricatti equation in lqr.m, and Jx+ρJu is computed by summing the two separate solutions for Jx and Ju. My current guess is that this is numerical error, although this seems way to big. Extra credit for anyone who comes up with an alternate explanation (like a typo at left!)

PROBLEM 2 See attached.



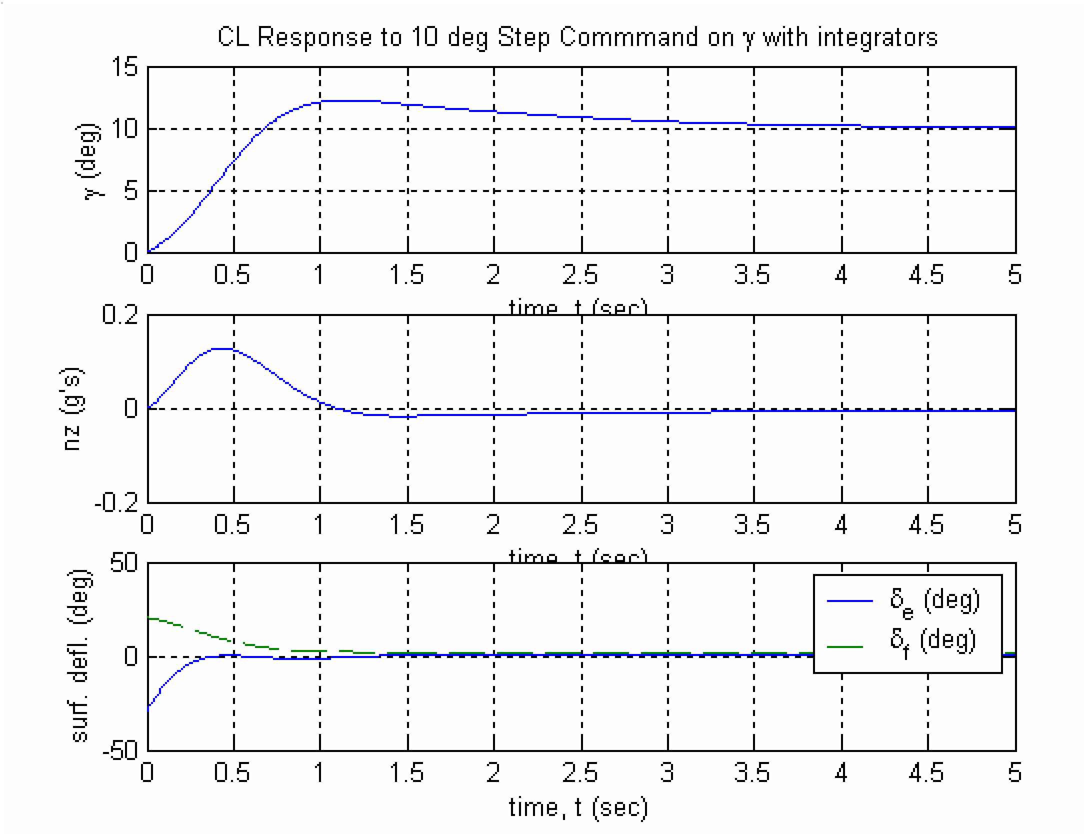
$$u = (M_u + KM_x)\,y_{des} - Kx - K_I x_I$$

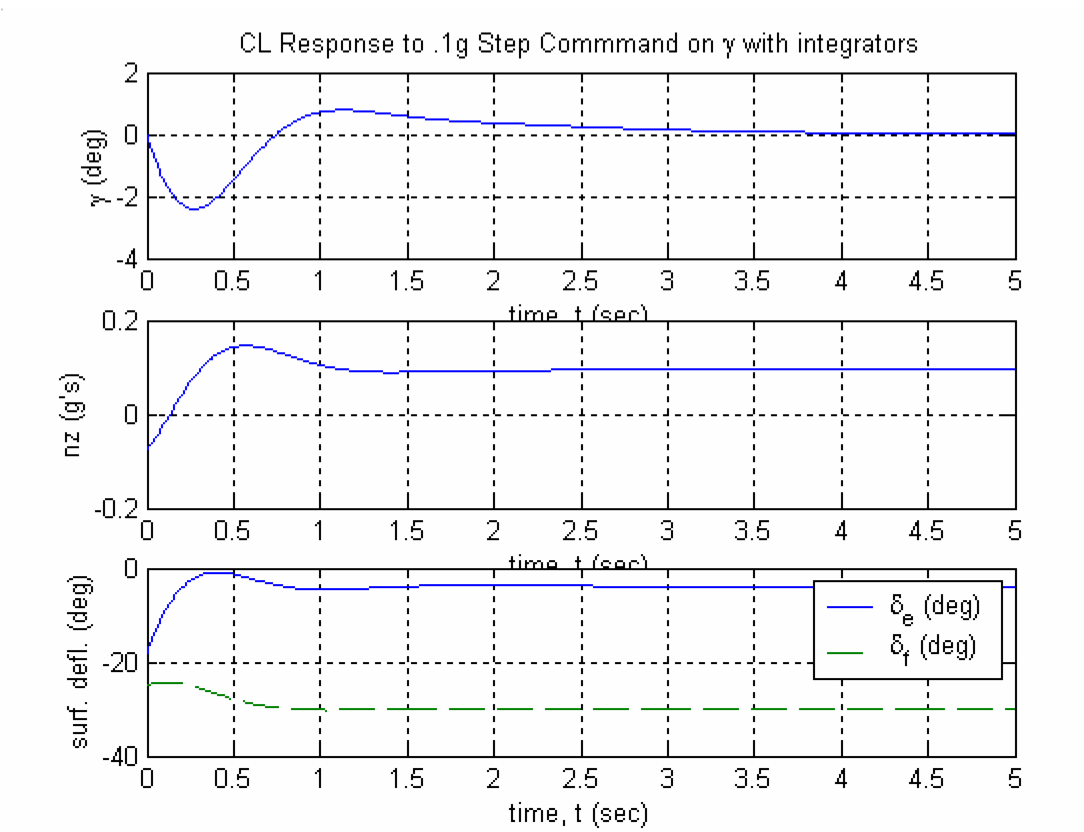
$$\dot{x}_I = -y_{des} + y = -y_{des} + (C_I x + D_I u)$$

Final Equations

$$\begin{bmatrix} \dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}\begin{bmatrix} x_I \\ x \end{bmatrix} + \begin{bmatrix} D \\ B \end{bmatrix} u + \begin{bmatrix} -I \\ 0 \end{bmatrix} y_{des}$$

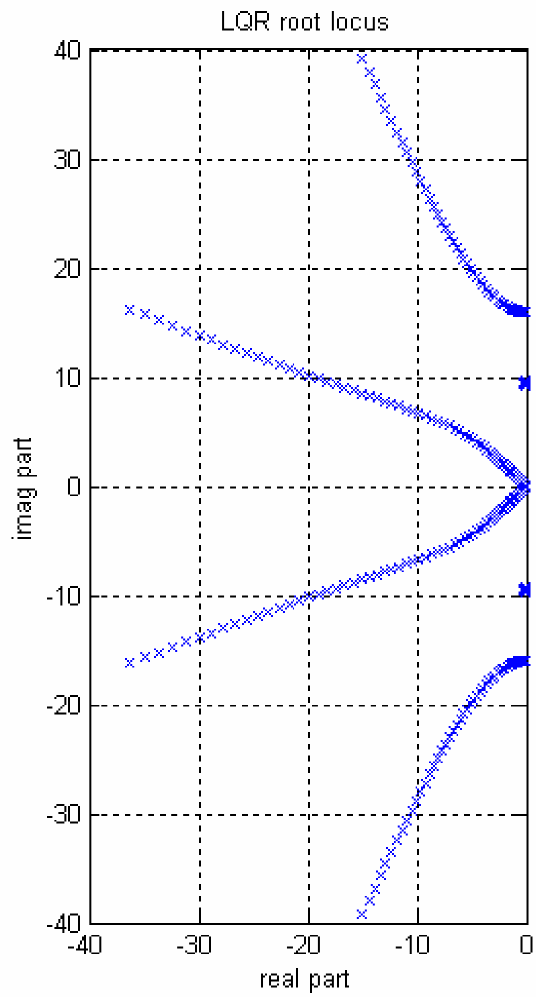
$$u = -\begin{bmatrix} K_I & K \end{bmatrix}\begin{bmatrix} x_I \\ x \end{bmatrix} + (M_u + KM_x)\,y_{des}$$


CL Response to 10 deg Step Commmand on γ with integrators
γ (deg)
nz (g's)
surf. defl. (deg)
time, t (sec)
δe (deg)
δf (deg)


CL Response to .1g Step Commmand on γ with integrators
γ (deg)
nz (g's)
surf. defl. (deg)
time, t (sec)
δe (deg)
δf (deg)

```
nz_com  = .1;                  % input nz magnitude
gam_com = 10*pi/180;     % input gamma

% System Matrices states are (q u alpha theta)
A = [-0.8  -0.0344   -12        0
      0   -0.014    -0.2904   -0.562
      1   -0.0057   -1.5       0
      1    0         0         0 ];

 B = [-19      -2.5
     -0.0115  -0.0087
     -0.16    -0.6
      0        0    ];

 C = [ 0   0   -1      1
      0   0    0.733  0 ];

 D = [ 0       0
      0.0768  .1134 ];

% (a) Compute Feedforward matrices
 M = [ A  B
      C D ];
 M = inv(M);
 Mx = M(1:4,5:6);
 Mu = M(5:6,5:6);

% (a) Create O.L. system, simulate steps
%
sys=ss(A,B*Mu,C,D*Mu);
T = [0:.01:500]';

U = [gam_com*ones(length(T),1)
0*nz_com*ones(length(T),1)];
y = lsim(sys,U,T,zeros(4,1));
figure(1)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('OL Response to 10 deg Step Commmand on
\gamma')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g''s)'), grid
dedf=(Mu*U')'*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'), grid
legend('\delta_e (deg)','\delta_f (deg)')
% ---

U = [0*gam_com*ones(length(T),1)
nz_com*ones(length(T),1)];
y = lsim(sys,U,T,zeros(4,1));
figure(2)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('OL Response to 0.1g Step Command on n_z')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g''s)'), grid
dedf=(Mu*U')'*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'), grid
legend('\delta_e (deg)','\delta_f (deg)')

% (b) Append system with integral of outputs
Aa = [zeros(2,2) C
     zeros(4,2) A];
Ba = [D;B];
Ca = [zeros(2,2),C];
Da = D;

% (c) Augmented system LQR (note N matrix)
%
Cq=[[eye(2,2),zeros(2,4)];Ca];
Dq=[zeros(2,2); Da];
Qy= diag([180/pi 1 180/pi 25]);
Q = Cq'*Qy*Cq;
R = Dq'*Qy*Dq + diag([10 10]);
N = Cq'*Qy*Dq;

K = lqr(Aa,Ba,Q,R,N);
Kc = K(1:2,3:6);
Ki = K(1:2,1:2);

% Final state space system with appropriate
% B matrix (see solution notes)
sys = ss(Aa-Ba*K, Ba*(Mu+Kc*Mx)+[-eye(2);zeros(4,2)],
Ca-Da*K, Da*(Mu+Kc*Mx));
T = [0:.01:5]';

U = [gam_com*ones(length(T),1)
0*nz_com*ones(length(T),1)];
[y,t,x] = lsim(sys,U,T,zeros(6,1));
figure(3)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('CL Response to 10 deg Step ...')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g''s)'), grid
dedf=( ((Mu+Kc*Mx)*U')'-(K*x')' )*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'), grid
legend('\delta_e (deg)','\delta_f (deg)')
% ---

U = [0*gam_com*ones(length(T),1)
nz_com*ones(length(T),1)];
[y,t,x] = lsim(sys,U,T,zeros(6,1));
figure(4)
subplot(311), plot(T,180/pi*y(:,1))
xlabel('time, t (sec)'), ylabel('\gamma (deg)'), grid
title('CL Response to .1g Step Commmand on \gamma;
Q=C''[5 0;0 5]C, R=[10 0;0 10]')
subplot(312), plot(T,y(:,2))
xlabel('time, t (sec)'), ylabel('nz (g''s)'), grid
dedf=( ((Mu+Kc*Mx)*U')'-(K*x')' )*180/pi;
subplot(313), plot(T,dedf(:,1),T,dedf(:,2),'--');
xlabel('time, t (sec)'), ylabel('surf. defl. (deg)'), grid
legend('\delta_e (deg)','\delta_f (deg)')
```

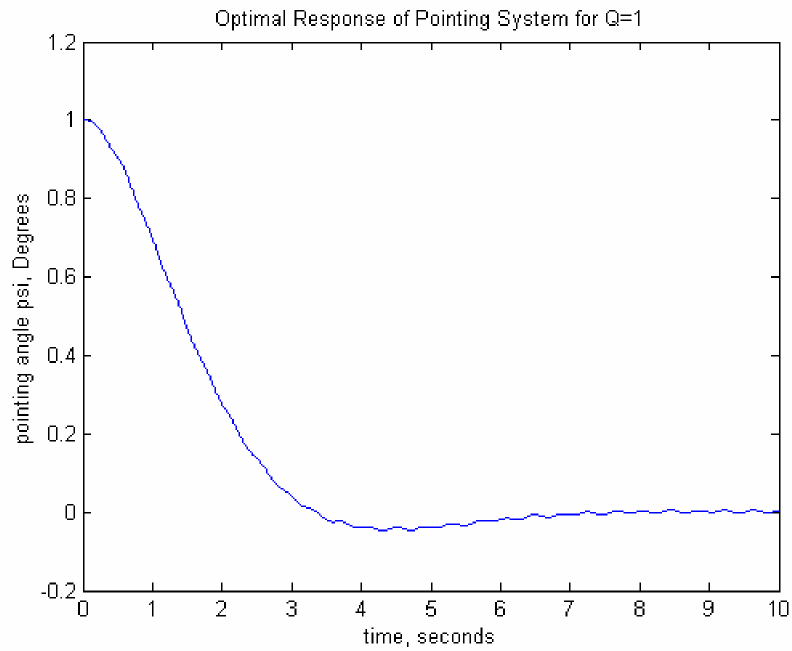
Problem 3(a)

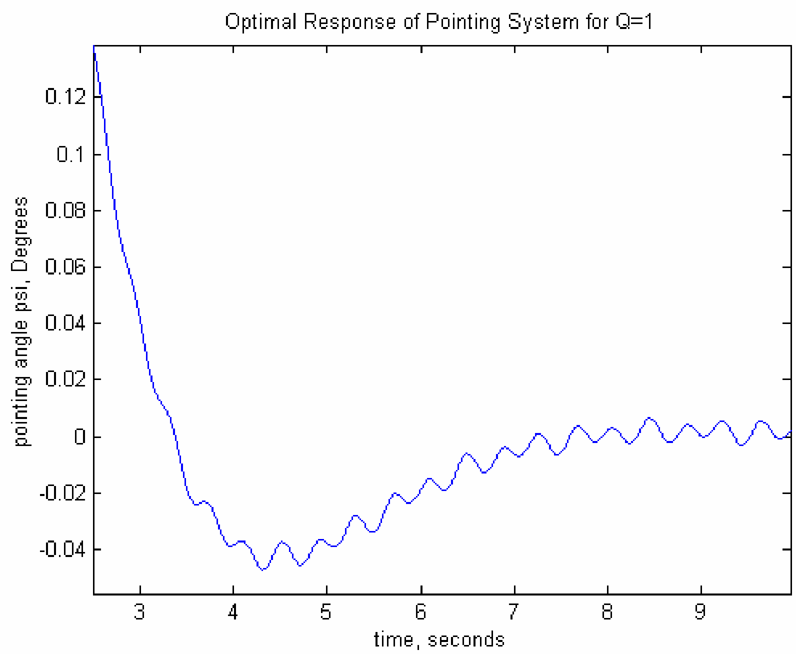



```
A1=[0 1;0 0];
A2=[0 1;-93.2 0];
A3=[0 1;-255 0];
Z=zeros(2,2);
A=[A1,Z,Z;Z,A2,Z;Z,Z,A3];
B=[0  1  0  0.1493  0  -1.1493]';
C=[1 0 1 0 1 0];

% (a) Since we did not study the root square locus,
%     this solution simply numerically solves for the
locus
qset=logspace(-4,8,100)
for Q=qset
    K=lqr(A,B,Q*C'*C,1);
    plot(eig(A-B*K)+.001*sqrt(-1),'x')
    hold on
end
axis('equal')
xlabel('real part'), ylabel('imag part'), title('LQR root
locus')

% (b) Time history fo Q=1
figure
K=lqr(A,B,C'*C,1);

figure(2)
t=[0:.001:10]';
u=zeros(size(t));
x0=[1 0 0 0 0 0]';

y=lsim(A-B*K,B,C,0,u,t,x0);
plot(t,y);
xlabel('time, seconds')
ylabel('pointing angle psi, Degrees')
title('Optimal Response of Pointing System for A=1')
```

Note that two of the poles proceed along ‘Butterworth patterns’, as described in class and in the handout notes. The remaining pair of poles are driven to a zero in the input-to-output transfer function, where the output was used to determine Q. Try setting Q=I instead of C’C to see a different root locus!

Problem 3(b) (the m-file shown on the previous page generates these plots).





Note that because one of the poles is being driven to a lightly damped zero, the pole's damping remains small. However, the residual at the output of interest is small, and therefore the pointing error (amplified in the second plot) is relatively small.