Topic #11

16.31 Feedback Control

State-Space Systems

- **State-space model features**
- Observability
- Controllability
- Minimal Realizations

State-Space Model Features

- There are some key characteristics of a state-space model that we need to identify.
  - Will see that these are very closely associated with the concepts of pole/zero cancellation in transfer functions.
- **Example:** Consider a simple system

$$G(s) = \frac{6}{s+2}$$

for which we develop the state-space model

$$\text{Model \# 1} \qquad \begin{aligned} \dot{x} &= -2x + 2u \\ y &= 3x \end{aligned}$$

- But now consider the new state space model $\bar{x} = [x \;\; x_2]^T$

$$\text{Model \# 2} \qquad \begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{x} \end{aligned}$$

which is **clearly different** than the first model.

- But let's looks at the transfer function of the new model:

$$\bar{G}(s) = C(sI - A)^{-1}B + D$$

- This is a bit strange, because previously our figure of merit when comparing one state-space model to another (page 8-8) was whether they reproduced the same same transfer function
  - Now we have two very different models that result in the same transfer function
  - Note that I showed the second model as having 1 extra state, but I could easily have done it with 99 extra states!!

- So what is going on?
  - The clue is that the dynamics associated with the second state of the model x_2 were eliminated when we formed the product

$$\bar{G}(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{s+2} \\ \frac{1}{s+1} \end{bmatrix}$$

  because the A is decoupled and there is a zero in the C matrix
  - Which is exactly the same as saying that there is a **pole-zero cancellation** in the transfer function $\tilde{G}(s)$

$$\frac{6}{s+2} = \frac{6(s+1)}{(s+2)(s+1)} \triangleq \tilde{G}(s)$$

  - Note that model #2 is one possible state-space model of $\tilde{G}(s)$ (has 2 poles)

- For this system we say that the dynamics associated with the second state are **unobservable** using this sensor (defines the C matrix).
  - There could be a lot "motion" associated with x_2, but we would be unware of it using this sensor.

- There is an analogous problem on the input side as well. Consider:

$$\text{Model \# 1} \qquad \begin{aligned} \dot{x} &= -2x + 2u \\ y &= 3x \end{aligned}$$

with $\bar{x} = [\,x \;\; x_2]^T$

$$\text{Model \# 3} \qquad \begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{x} \end{aligned}$$

which is also **clearly different** than model #1, and has a different form from the second model.

$$\begin{aligned} \hat{G}(s) &= \begin{bmatrix} 3 & 2 \end{bmatrix} \left(sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{s+2} & \frac{2}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2} \quad !! \end{aligned}$$

- Once again the dynamics associated with the pole at $s = -1$ are cancelled out of the transfer function.
  - But in this case it occurred because there is a 0 in the B matrix
- So in this case we can "see" the state x_2 in the output $C = \begin{bmatrix} 3 & 2 \end{bmatrix}$, but we cannot "influence" that state with the input since $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
- So we say that the dynamics associated with the second state are **uncontrollable** using this actuator (defines the B matrix).

- Of course it can get even worse because we could have

$$\begin{aligned}\dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{x}\end{aligned}$$

- So now we have

$$\begin{aligned}\widetilde{G(s)} &= \begin{bmatrix} 3 & 0 \end{bmatrix} \left(sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{s+2} & \frac{0}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2} \quad !!\end{aligned}$$

- Get same result for the transfer function, but now the dynamics associated with x_2 are both unobservable and uncontrollable.

- **Summary:**
  Dynamics in the state-space model that are **uncontrollable**, **unobservable**, or **both** do not show up in the transfer function.

- Would like to develop models that **only have** dynamics that are both **controllable** and **observable**
  $\Rrightarrow$ called a **minimal realization**
  - It is has the lowest possible order for the given transfer function.

- But first need to develop tests to determine if the models are observable and/or controllable

Observability

- **Definition:** An LTI **system** is **observable** if the initial state $x(0)$ can be uniquely deduced from the knowledge of the input $u(t)$ and output $y(t)$ for all t between 0 and any $T > 0$.
  - If $x(0)$ can be deduced, then we can reconstruct $x(t)$ exactly because we know $u(t) \Rrightarrow$ we can find $x(t)$ $\forall\, t$.
  - Thus we need only consider the zero-input (homogeneous) solution to study observability.

$$y(t) = Ce^{At}x(0)$$

- This definition of observability is consistent with the notion we used before of being able to "see" all the states in the output of the decoupled examples
  - ROT: For those decoupled examples, if part of the state cannot be "seen" in $y(t)$, then it would be impossible to deduce that part of $x(0)$ from the outputs $y(t)$.

- **Definition:** A **state** $x^\star \neq 0$ is said to be **unobservable** if the zero-input solution $y(t)$, with $x(0) = x^\star$, is zero for all $t \geq 0$
  - Equivalent to saying that $x^\star$ is an unobservable state if

$$Ce^{At}x^\star = 0 \ \forall \ t \ \geq 0$$

- For the problem we were just looking at, consider Model #2 with $x^\star = [\ 0 \quad 1\ ]^T \neq 0$, then

$$\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{x} \end{aligned}$$

so

$$\begin{aligned} Ce^{At}x^\star &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \ \forall \ t \end{aligned}$$

So, $x^\star = [\ 0 \quad 1\ ]^T$ is an unobservable state for this system.

- But that is as expected, because we knew there was a problem with the state x_2 from the previous analysis

- **Theorem: An LTI system is observable iff it has no unobservable states.**
  - We normally just say that the **pair (A,C) is observable.**

- **Pseudo-Proof:** Let $x^\star \neq 0$ be an unobservable state and compute the outputs from the initial conditions $x_1(0)$ and $x_2(0) = x_1(0) + x^\star$
  - Then

$$y_1(t) = Ce^{At}x_1(0) \quad \text{and} \quad y_2(t) = Ce^{At}x_2(0)$$

but

  - Thus 2 different initial conditions give the same output $y(t)$, so it would be impossible for us to deduce the actual initial condition of the system $x_1(t)$ or $x_2(t)$ given $y_1(t)$

- Testing system observability by searching for a vector $x(0)$ such that $Ce^{At}x(0) = 0 \; \forall \; t$ is feasible, but very hard in general.
  - Better tests are available.

- **Theorem: The vector $x^\star$ is an unobservable state if**

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x^\star = 0$$

- **Pseudo-Proof:** If $x^\star$ is an unobservable state, then by definition,

$$Ce^{At}x^\star = 0 \quad \forall\, t \geq 0$$

But all the derivatives of Ce^{At} exist and for this condition to hold, all derivatives must be zero at $t = 0$. Then

$$Ce^{At}x^\star\Big|_{t=0} = 0 \;\Rightarrow\; Cx^\star = 0$$

$$\frac{d}{dt}Ce^{At}x^\star\Big|_{t=0} = 0 \;\Rightarrow$$

$$\frac{d^2}{dt^2}Ce^{At}x^\star\Big|_{t=0} = 0 \;\Rightarrow$$

$$\vdots$$

$$\frac{d^k}{dt^k}Ce^{At}x^\star\Big|_{t=0} = 0 \;\Rightarrow$$

- We only need retain up to the $n-1^{\text{th}}$ derivative because of the *Cayley-Hamilton* theorem.

- **Simple test:** Necessary and sufficient condition for observability is that

$$\texttt{rank}\ \mathcal{M}_o \triangleq \texttt{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

- Why does this make sense?
  - The requirement for an unobservable state is that for $x^\star \neq 0$

$$\mathcal{M}_o x^\star = 0$$

  - Which is equivalent to saying that $x^\star$ is orthogonal to each row of $\mathcal{M}_o$.
  - But if the rows of $\mathcal{M}_o$ are considered to be vectors and these **span the full n-dimensional space**, then it is not possible to find an n-vector $x^\star$ that is orthogonal to each of these.
  - To determine if the n rows of $\mathcal{M}_o$ span the full n-dimensional space, we need to test their **linear independence**, which is equivalent to the `rank` test[1].

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[1]Let M be a $m \times p$ matrix, then the `rank` of M satisfies:

1. `rank` $M \equiv$ number of linearly independent columns of M
2. `rank` $M \equiv$ number of linearly independent rows of M
3. `rank` $M \leq \min\{m, p\}$