Topic #12

16.31 Feedback Control

State-Space Systems

- **State-space model features**
- Controllability

Controllability

- **Definition:** An LTI **system** is **controllable** if, for every $x^\star(t)$ and every $T > 0$, there exists an input function $u(t)$, $0 < t \leq T$, such that the system state goes from $x(0) = 0$ to $x(T) = x^\star$.

  – Starting at 0 is not a special case – if we can get to any state in finite time from the origin, then we can get from any initial condition to that state in finite time as well.

  – Need only consider the forced solution to study controllability.

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

  – Change of variables $\tau_2 = t - \tau$, $d\tau = -d\tau_2$ gives

$$x(t) = \int_0^t e^{A\tau_2} Bu(t - \tau_2) d\tau_2$$

- This definition of observability is consistent with the notion we used before of being able to "influence" all the states in the system in the decoupled examples we looked at before.

  – ROT: For those decoupled examples, if part of the state cannot be "influenced" by $u(t)$, then it would be impossible to move that part of the state from 0 to $x^\star$

- **Definition:** A **state** $x^\star \neq 0$ is said to be **uncontrollable** if the forced state response $x(t)$ is orthogonal to $x^\star$ $\forall\ t > 0$ and all input functions.
  - "You cannot get there from here"

- This is equivalent to saying that $x^\star$ is an uncontrollable state if

$$(x^\star)^T \int_0^t e^{A\tau_2} B u(t-\tau_2) d\tau_2$$

$$= \int_0^t (x^\star)^T e^{A\tau_2} B u(t-\tau_2) d\tau_2 = 0$$

- Since this identity must hold for all input functions $u(t-\tau_2)$, this can only be true if

$$(x^\star)^T e^{At} B \equiv 0 \ \ \forall\ t \geq 0$$

- For the problem we were just looking at, consider Model #3 with $x^\star = [\ 0 \ \ 1\ ]^T \neq 0$, then

$$\text{Model \# 3} \qquad \dot{\bar{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{x}$$

so

$$(x^\star)^T e^{At} B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0 \quad \forall\, t$$

So $x^\star = [\ 0 \ \ 1\ ]^T$ is an uncontrollable state for this system.

- But that is as expected, because we knew there was a problem with the state x_2 from the previous analysis

- **Theorem: An LTI system is controllable iff it has no uncontrollable states.**
  - We normally just say that the **pair (A,B) is controllable.**

  Pseudo-Proof: The theorem essentially follows by the definition of an uncontrollable state.
  - If you had an uncontrollable state $x^\star$, then it is orthogonal to the forced response state $x(t)$, which means that the system cannot *reach* it in finite time $\rightsquigarrow$ the system would be uncontrollable.

- **Theorem: The vector $x^\star$ is an uncontrollable state iff**

$$(x^\star)^T \left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array} \right] = 0$$

  - See page 81.

- **Simple test:** Necessary and sufficient condition for controllability is that

$$\texttt{rank}\, \mathcal{M}_c \triangleq \texttt{rank} \left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array} \right] = n$$

$$\text{With Model \# 2:} \quad \begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{x} \end{aligned}$$

$$\mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$$

– `rank` $\mathcal{M}_0 = 1$ and `rank` $\mathcal{M}_c = 2$

– So this model of the system is controllable, but not observable.

$$\text{With Model \# 3:} \quad \begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{x} \end{aligned}$$

$$\mathcal{M}_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -6 & -2 \end{bmatrix}$$

$$\mathcal{M}_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}$$

– `rank` $\mathcal{M}_0 = 2$ and `rank` $\mathcal{M}_c = 1$

– So this model of the system is observable, but not controllable.

- Note that controllability/observability are **not** intrinsic properties of a system. Whether the model has them or not depends on the representation that you choose.
  – But they indicate that something else more fundamental is wrong...

Example: Loss of Observability

- Typical scenario: consider system $G(s)$ of the form

$$u \rightarrow \boxed{\frac{1}{s+a}}_{x_1} \rightarrow \boxed{\frac{s+a}{s+1}}_{x_2} \rightarrow y$$

so that

$$G(s) = \frac{s+a}{s+1} \cdot \frac{1}{s+a}$$

- Clearly a pole-zero cancelation in this system (pole $s = -a$)

- The state space model for the system is:

$$\begin{aligned} \dot{x}_1 &= -ax_1 + u \\ \dot{x}_2 &= -x_2 + (a-1)x_2 \\ y &= x_1 + x_2 \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} -a & 0 \\ a-1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix},\ D = 0$$

- The Observability/Controllability tests are ($a = 2$):

$$\texttt{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = 1 < n = 2$$

$$\texttt{rank} \begin{bmatrix} B & AB \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 2$$

- System controllable, but unobservable. Consistent with the picture:
  - Both states can be influenced by u
  - But e^{-at} mode dynamics canceled out of the output by the zero.

Example: Loss of Controllability

- Repeat the process, but now use the system $G(s)$ of the form

$$u \rightarrow \boxed{\frac{s+a}{s+1}}_{x_2} \rightarrow \boxed{\frac{1}{s+a}}_{x_1} \rightarrow y$$

  so that

$$G(s) = \frac{1}{s+a} \cdot \frac{s+a}{s+1}$$

- Still a pole-zero cancelation in this system (pole $s = -a$)
- The state space model for the system is:

$$\begin{aligned} \dot{x}_1 &= -ax_1 + x_2 + u \\ \dot{x}_2 &= -x_2 + (a-1)u \\ y &= x_1 \end{aligned}$$

$$\Rightarrow \; A_2 = \begin{bmatrix} -a & 1 \\ 0 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ a-1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_2 = 0$$

- The Observability/Controllability tests are ($a = 2$):

$$\texttt{rank} \begin{bmatrix} C_2 \\ C_2 A_2 \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = 2$$

$$\texttt{rank} \begin{bmatrix} B_2 & A_2 B_2 \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 1 < n = 2$$

- System observable, but uncontrollable. Consistent with the picture:
  - u can influence state x_2, but effect on x_1 canceled by zero
  - Both states can be seen in the output (x_1 directly, and x_2 because it drives the dynamics associated with x_1)

Modal Tests

- Earlier examples showed the relative simplicity of testing observability/controllability for system with a *decoupled* A matrix.

- There is, of course, a very special decoupled form for the state-space model: the **Modal Form** (8–5)

- Assuming that we are given the model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

and the A is diagonalizable ($A = T\Lambda T^{-1}$) using the transformation

$$T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

based on the eigenvalues of A. Note that we wrote:

$$T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

which is a column of rows.

- Then define a new state so that $x = Tz$, then

$$\begin{aligned}\dot{z} &= T^{-1}\dot{x} = T^{-1}(Ax + Bu) = (T^{-1}AT)z + T^{-1}Bu \\ &= \Lambda z + T^{-1}Bu \\ y &= Cx + Du = CTz + Du\end{aligned}$$

- The new model in the state z is diagonal. There is no coupling in the dynamics matrix Λ.

- But by definition,

$$T^{-1}B = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} B$$

  and

$$CT = C \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

- Thus if it turned out that

$$w_i^T B \equiv 0$$

  then that element of the state vector z_i would be **uncontrollable** by the input u.

- Also, if

$$C v_j \equiv 0$$

  then that element of the state vector z_j would be **unobservable** with this sensor.

- Thus, **all modes of the system are controllable and observable** if it can be shown that

$$w_i^T B \neq 0 \;\; \forall \; i$$

  and

$$C v_j \neq 0 \;\; \forall \; j$$

Cancelation

- Examples show the close connection between pole-zero cancelation and loss of observability and controllability. Can be strengthened.

- **Theorem:** The mode (λ_i, v_i) of a system (A, B, C, D) is unobservable iff the system has a zero at λ_i with direction $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$.

- **Proof:** If the system is unobservable at λ_i, then we know

$$\begin{aligned} (\lambda_i I - A)v_i &= 0 \quad \text{It is a mode} \\ Cv_i &= 0 \quad \text{That mode is unobservable} \end{aligned}$$

  Combine to get:

$$\begin{bmatrix} (\lambda_i I - A) \\ C \end{bmatrix} v_i = 0$$

  Or

$$\begin{bmatrix} (\lambda_i I - A) & -B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ 0 \end{bmatrix} = 0$$

  which implies that the system has a zero at that frequency as well, with direction $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$.

- Can repeat the process looking for loss of controllability, but now using zeros with left direction $\begin{bmatrix} w_i^T & 0 \end{bmatrix}$.

- **Combined Definition:** when a MIMO zero causes loss of either observability or controllability we say that there is a pole/zero cancelation.

  – MIMO pole-zero (right direction generalized eigenvector) cancelation $\Leftrightarrow$ mode is unobservable

  – MIMO pole-zero (left direction generalized eigenvector) cancelation $\Leftrightarrow$ mode is uncontrollable

- **Note:** This cancelation requires an agreement of both the frequency and the directionality of the system mode (eigenvector) and zero $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$ or $\begin{bmatrix} w_i^T & 0 \end{bmatrix}$.

Weaker Conditions

- Often it is too much to assume that we will have full observability and controllability. Often have to make do with the following:

- A system is called **detectable** if all unstable modes are **observable**

- A system is called **stabilizable** if all unstable modes are **controllable**

- So if you had a stabilizable and detectable system, there could be dynamics that you are not aware of and cannot influence, but you know that they are at least stable.