# Topic #13

16.31 Feedback Control

**State-Space Systems**

- **Full-state Feedback Control**
- How do we change the poles of the state-space system?
- Or, even if we can change the pole locations.
- Where do we change the pole locations to?
- How well does this approach work?

# Full-state Feedback Controller

- Assume that the single-input system dynamics are given by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

so that $D = 0$.

  - The multi-actuator case is quite a bit more complicated as we would have many extra degrees of freedom.

- Recall that the system poles are given by the eigenvalues of $A$.

  - Want to use the input $u(t)$ to modify the eigenvalues of $A$ to change the system dynamics.

- Assume a full-state feedback of the form:

$$u = r - Kx$$

where $r$ is some **reference input** and the **gain** $K$ is $\mathcal{R}^{1\times n}$

  - If $r = 0$, we call this controller a **regulator**

- Find the closed-loop dynamics:

$$\begin{aligned}\dot{x} &= Ax + B(r - Kx) \\ &= (A - BK)x + Br \\ &= A_{cl}x + Br \\ y &= Cx\end{aligned}$$

- **Objective:** Pick $K$ so that $A_{cl}$ has the desired properties, *e.g.,*
  - $A$ unstable, want $A_{cl}$ stable
  - Put 2 poles at $-2 \pm 2j$

- Note that there are $n$ parameters in $K$ and $n$ eigenvalues in $A$, so it looks promising, but what can we achieve?

- **Example #1:** Consider:

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

  - Then

$$\det(sI - A) = (s-1)(s-2) - 1 = s^2 - 3s + 1 = 0$$

  so the system is unstable.

  - Define $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x = -Kx$, then

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 1 & 2 \end{bmatrix}$$

  - So then we have that

$$\det(sI - A_{cl}) = s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2) = 0$$

  - Thus, by choosing $k_1$ and $k_2$, we can put $\lambda_i(A_{cl})$ anywhere in the complex plane (assuming complex conjugate pairs of poles).

- To put the poles at $s = -5, \ -6$, compare the *desired characteristic equation*

$$(s+5)(s+6) = s^2 + 11s + 30 = 0$$

with the closed-loop one

$$s^2 + (k_1 - 3)x + (1 - 2k_1 + k_2) = 0$$

to conclude that

$$\left.\begin{array}{c} k_1 - 3 = 11 \\ 1 - 2k_1 + k_2 = 30 \end{array}\right\} \quad \begin{array}{l} k_1 = 14 \\ k_2 = 57 \end{array}$$

so that $K = \left[\ 14 \ \ 57\ \right]$, which is called **Pole Placement**.

- Of course, it is not always this easy, as the issue of **controllability** must be addressed.

- **Example #2:** Consider this system:

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

with the same control approach

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[\ k_1 \ \ k_2\ \right] = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix}$$

so that

$$\det(sI - A_{cl}) = (s - 1 + k_1)(s - 2) = 0$$

So the feedback control can modify the pole at $s = 1$, but it cannot move the pole at $s = 2$.

- **The system cannot be stabilized with full-state feedback control.**

- What is the reason for this problem?
  – It is associated with loss of controllability of the $e^{2t}$ mode.

- Consider the basic controllability test:

$$\mathcal{M}_c = \left[\, B \,\middle|\, AB \,\right] = \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$$

  So that $\texttt{rank}\ \mathcal{M}_c = 1 < 2$.

- Consider the **modal test** to develop a little more insight:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \text{decompose as} \quad AV = V\Lambda \quad \Rightarrow \Lambda = V^{-1}AV$$

  where

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

  Convert

$$\dot{x} = Ax + Bu \quad \xrightarrow{z = V^{-1}x} \quad \dot{z} = \Lambda z + V^{-1}Bu$$

  where $z = \left[\, z_1 \;\; z_2 \,\right]^T$. But:

$$V^{-1}B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$$

  so that the dynamics in modal form are:

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} u$$

- With this zero in the modal $B$-matrix, can easily see that the mode associated with the $z_2$ state is **uncontrollable.**
  – **Must assume that the pair $(A,\ B)$ are controllable.**

# Ackermann's Formula

- The previous outlined a design procedure and showed how to do it by hand for second-order systems.
  - Extends to higher order (controllable) systems, but tedious.

- **Ackermann's Formula** gives us a method of doing this entire design process is one easy step.

$$K = \left[\, 0 \;\; \ldots \;\; 0 \;\; 1 \,\right] \mathcal{M}_c^{-1} \Phi_d(A)$$

  where
  - $\mathcal{M}_c = \left[\, B \;\; AB \;\; \ldots \;\; A^{n-1}B \,\right]$
  - $\Phi_d(s)$ is the characteristic equation for the closed-loop poles, which we then evaluate for $s = A$.
  - It is explicit that the **system must be controllable** because we are inverting the controllability matrix.

- Revisit **example # 1:** $\Phi_d(s) = s^2 + 11s + 30$

$$\mathcal{M}_c = \left[\, B \,\middle|\, AB \,\right] = \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

  So

$$\begin{aligned} K &= \left[\, 0 \;\; 1 \,\right] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 + 11 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 30I \right) \\ &= \left[\, 0 \;\; 1 \,\right] \left( \begin{bmatrix} 43 & 14 \\ 14 & 57 \end{bmatrix} \right) = \left[\, 14 \;\; 57 \,\right] \end{aligned}$$

- Automated in Matlab: `place.m` & `acker.m` (see `polyvalm.m` too)

- Where did this formula come from?

- For simplicity, consider a third-order system (case #2), but this extends to any order.

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

  – See key benefit of using **control canonical** state-space model

  – This form is useful because the characteristic equation for the system is obvious $\Rightarrow \det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$

- Can show that

$$\begin{aligned} A_{cl} = A - BK &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \\ &= \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

  so that the characteristic equation for the system is still obvious:

$$\begin{aligned} \Phi_{cl}(s) &= \det(sI - A_{cl}) \\ &= s^3 + (a_1 + k_1)s^2 + (a_2 + k_2)s + (a_3 + k_3) = 0 \end{aligned}$$

- We then compare this with the desired characteristic equation developed from the desired closed-loop pole locations:

$$\Phi_d(s) = s^3 + (\alpha_1)s^2 + (\alpha_2)s + (\alpha_3) = 0$$

to get that

$$\left.\begin{array}{c} a_1 + k_1 = \alpha_1 \\ \vdots \\ a_n + k_n = \alpha_n \end{array}\right\} \begin{array}{c} k_1 = \alpha_1 - a_1 \\ \vdots \\ k_n = \alpha_n - a_n \end{array}$$

- To get the specifics of the Ackermann formula, we then:
  - Take an arbitrary $A, B$ and transform it to the control canonical form ($x \rightsquigarrow z = T^{-1}x$)
  - Solve for the gains $\hat{K}$ using the formulas above for the state $z$ ($u = \hat{K}z$)
  - Then switch back to gains needed for the state $x$, so that

$$K = \hat{K}T^{-1}$$

  ($u = \hat{K}z = Kx$)

- Pole placement is a very powerful tool and we will be using it for most of this course.