Topic #17

16.31 Feedback Control

State-Space Systems

- Closed-loop control using estimators and regulators.
- Dynamics output feedback

- "Back to reality"

Combined Estimators and Regulators

- Can now evaluate the stability and/or performance of a controller when we **design** K assuming that $u = -Kx$, but we **implement**

$$u = -K\hat{x}$$

- Assume that we have designed a closed-loop estimator with gain L

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y - \hat{y}) \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned}$$

- Then we have that the closed-loop system dynamics are given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y - \hat{y}) \\ y(t) &= Cx(t) \\ \hat{y}(t) &= C\hat{x}(t) \\ u &= -K\hat{x} \end{aligned}$$

- Which can be compactly written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad \Rightarrow \quad \dot{x}_{cl} = A_{cl} x_{cl}$$

- This does not look too good at this point – not even obvious that the closed-system is stable.

$$\lambda_i(A_{cl}) = ??$$

- Can fix this problem by introducing a new variable $\tilde{x} = x - \hat{x}$ and then converting the closed-loop system dynamics using the *similarity transformation* T

$$\tilde{x}_{cl} \triangleq \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = T x_{cl}$$

  – Note that $T = T^{-1}$

- Now rewrite the system dynamics in terms of the state $\tilde{x}_{cl}$

$$A_{cl} \quad \Rightarrow \quad T A_{cl} T^{-1} \triangleq \bar{A}_{cl}$$

  – Note that similarity transformations preserve the eigenvalues, so we are guaranteed that

$$\lambda_i(A_{cl}) \equiv \lambda_i(\bar{A}_{cl})$$

- Work through the math:

$$\begin{aligned} \bar{A}_{cl} &= \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} A & -BK \\ A - LC & -A + LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \end{aligned}$$

- Because $\bar{A}_{cl}$ is block upper triangular, we know that the closed-loop poles of the system are given by

$$\det(sI - \bar{A}_{cl}) \triangleq \det(sI - (A - BK)) \cdot \det(sI - (A - LC)) = 0$$

- **Observation: The closed-loop poles for this system consist of the union of the regulator poles and estimator poles.**

- So we can just design the estimator/regulator **separately** and combine them at the end.
  - Called the **Separation Principle**.
  - Just keep in mind that the pole locations you are picking for these two sub-problems will also be the closed-loop pole locations.

- **Note:** the separation principle means that there will be **no** ambiguity or uncertainty about the stability and/or performance of the closed-loop system.
  - The closed-loop poles will be exactly where you put them!!
  - And we have not even said what compensator does this amazing accomplishment!!!

The Compensator

- **Dynamic Output Feedback Compensator** is the combination of the regulator and estimator using $u = -K\hat{x}$

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y - \hat{y}) \\ &= A\hat{x}(t) - BK\hat{x} + L(y - C\hat{x})\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \dot{\hat{x}}(t) &= (A - BK - LC)\hat{x}(t) + Ly \\ u &= -K\hat{x}\end{aligned}$$

- Rewrite with new state $x_c \equiv \hat{x}$

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y \\ u &= -C_c x_c\end{aligned}$$

where the **compensator dynamics** are given by:

$$A_c \triangleq A - BK - LC \;, \quad B_c \triangleq L \;, \quad C_c \triangleq K$$

  – Note that the compensator maps *sensor measurements* to *actuator commands*, as expected.

- Closed-loop system stable if regulator/estimator poles placed in the LHP, but compensator dynamics do not need to be stable.

$$\lambda_i(A - BK - LC) = ??$$

- For consistency in the implementation with the classical approaches, define the **compensator transfer function** so that

$$u = -G_c(s)y$$

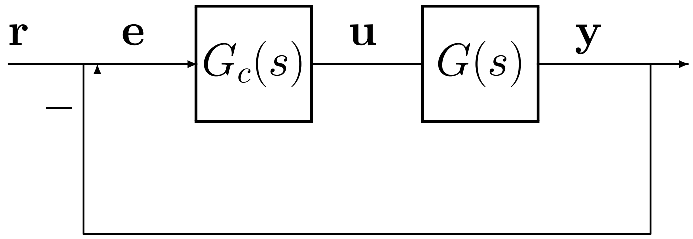
  – From the state-space model of the compensator:

$$\begin{aligned}\frac{U(s)}{Y(s)} &\triangleq -G_c(s) \\ &= -C_c(sI - A_c)^{-1}B_c \\ &= -K(sI - (A - BK - LC))^{-1}L\end{aligned}$$

$$\Rightarrow \quad \mathbf{G_c(s) = C_c(sI - A_c)^{-1}B_c}$$

- Note that it is often very easy to provide classical interpretations (such as lead/lag) for the compensator $G_c(s)$.

- One way to implement this compensator with a reference command $r(t)$ is to change the feedback to be on $e(t) = r(t) - y(t)$ rather than just $-y(t)$



$$\Rightarrow \quad u = G_c(s)e = G_c(s)(r - y)$$

  – So we still have $u = -G_c(s)y$ if $r = 0$.

  – Intuitively appealing because it is the **same approach** used for the classical control, but it turns out not to be the best approach. More on this later.

Mechanics

- Basics:

$$e = r - y, \qquad u = G_c e, \qquad y = Gu$$

$$G_c(s): \qquad \dot{x}_c = A_c x_c + B_c e \;\; , \quad u = C_c x_c$$

$$G(s): \qquad \dot{x} = Ax + Bu \quad , \quad y = Cx$$

- Loop dynamics $L = G_c(s)G(s) \quad \Rightarrow \quad y = L(s)e$

$$\begin{aligned} \dot{x} &= Ax + BC_c x_c \\ \dot{x}_c &= + A_c x_c + B_c e \end{aligned}$$

$$L(s) \quad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} e$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}$$

- To "close the loop", note that $e = r - y$, then

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \left(r - \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \right) \\ &= \begin{bmatrix} A & BC_c \\ -B_c C & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \end{aligned}$$

  – A_{cl} is not exactly the same as on page 17-1 because we have re-arranged where the negative sign enters into the problem. Same result though.

Simple Example

- Let $G(s) = 1/s^2$ with state-space model given by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

- Design the regulator to place the poles at $s = -4 \pm 4j$

$$\lambda_i(A - BK) = -4 \pm 4j \quad \Rightarrow K = \begin{bmatrix} 32 & 8 \end{bmatrix}$$

  – Time constant of regulator poles $\tau_c = 1/\zeta\omega_n \approx 1/4 = 0.25$ sec

- Put estimator poles so that the time constant is faster $\tau_e \approx 1/10$

  – Use real poles, so $\Phi_e(s) = (s+10)^2$

$$\begin{aligned} L &= \Phi_e(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 + 20 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 100 & 20 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix} \end{aligned}$$

- Compensator:

$$\begin{aligned} A_c &= A - BK - LC \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 32 & 8 \end{bmatrix} - \begin{bmatrix} 20 \\ 100 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -20 & 1 \\ -132 & -8 \end{bmatrix} \end{aligned}$$

$$B_c = L = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$$

$$C_c = K = \begin{bmatrix} 32 & 8 \end{bmatrix}$$

- Compensator transfer function:

$$\begin{aligned} G_c(s) &= C_c(sI - A_c)^{-1}B_c \triangleq \frac{U}{E} \\ &= 1440\frac{s + 2.222}{s^2 + 28s + 292} \end{aligned}$$

- Note that the compensator has a low frequency real zero and two higher frequency poles.
  - Thus it looks like a "lead" compensator.

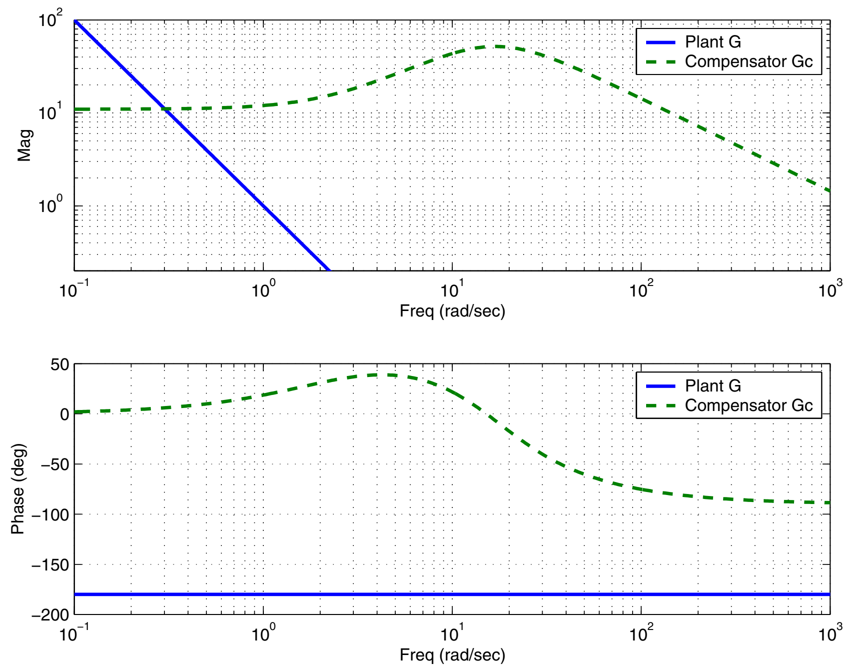


Figure 1: Plant is pretty simple and the compensator looks like a lead 2–10 rads/sec.

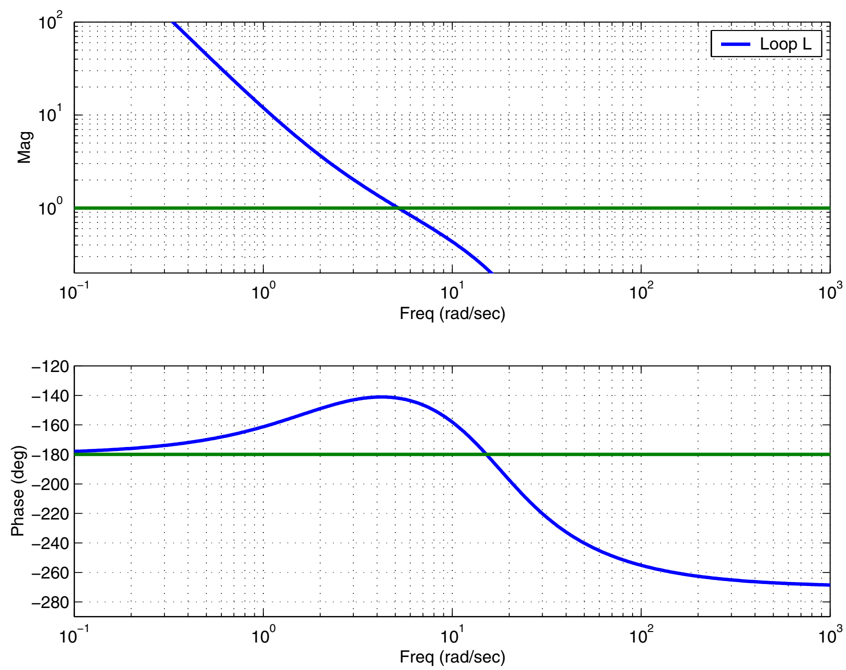


Figure 2: Loop transfer function $L(s)$ shows the slope change near $\omega_c = 5$ rad/sec. Note that we have a large PM and GM.

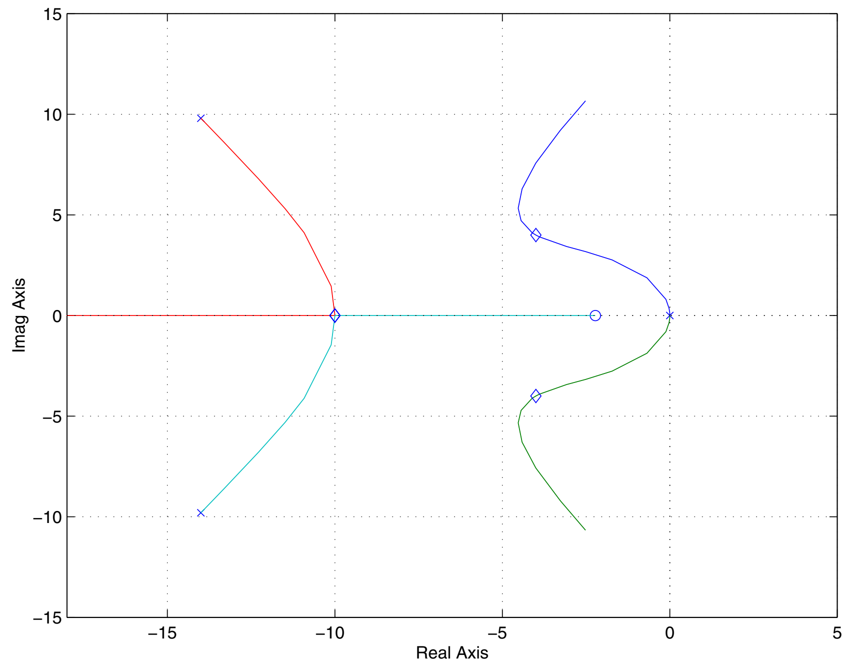


Figure 3: Freeze the compensator poles and zeros and look at the root locus of closed-loop poles versus an additional loop gain α (nominally $\alpha = 1$.)

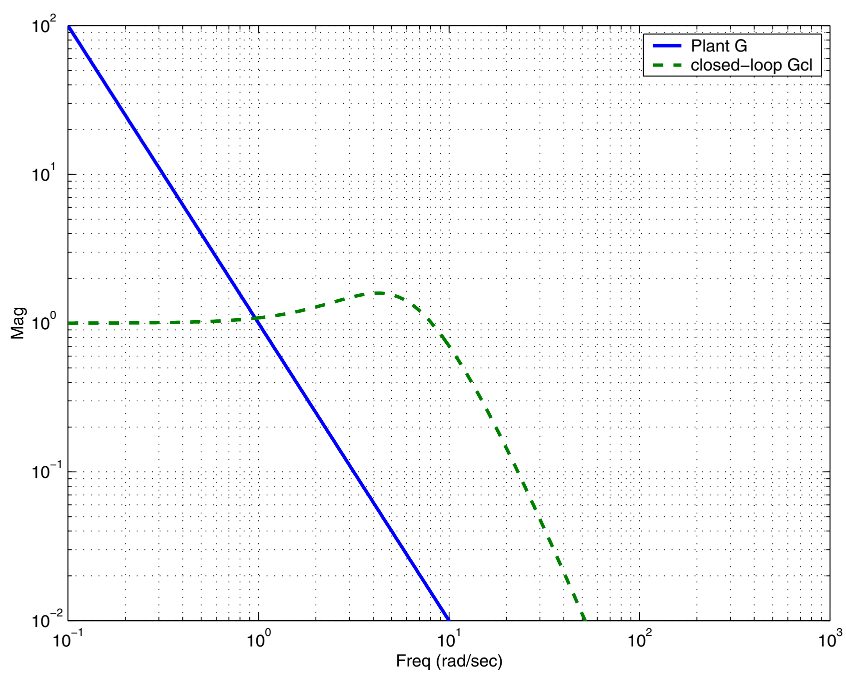


Figure 4: Closed-loop transfer function.

Figure 5: Example #1: $G(s) = \frac{8 \cdot 14 \cdot 20}{(s+8)(s+14)(s+20)}$

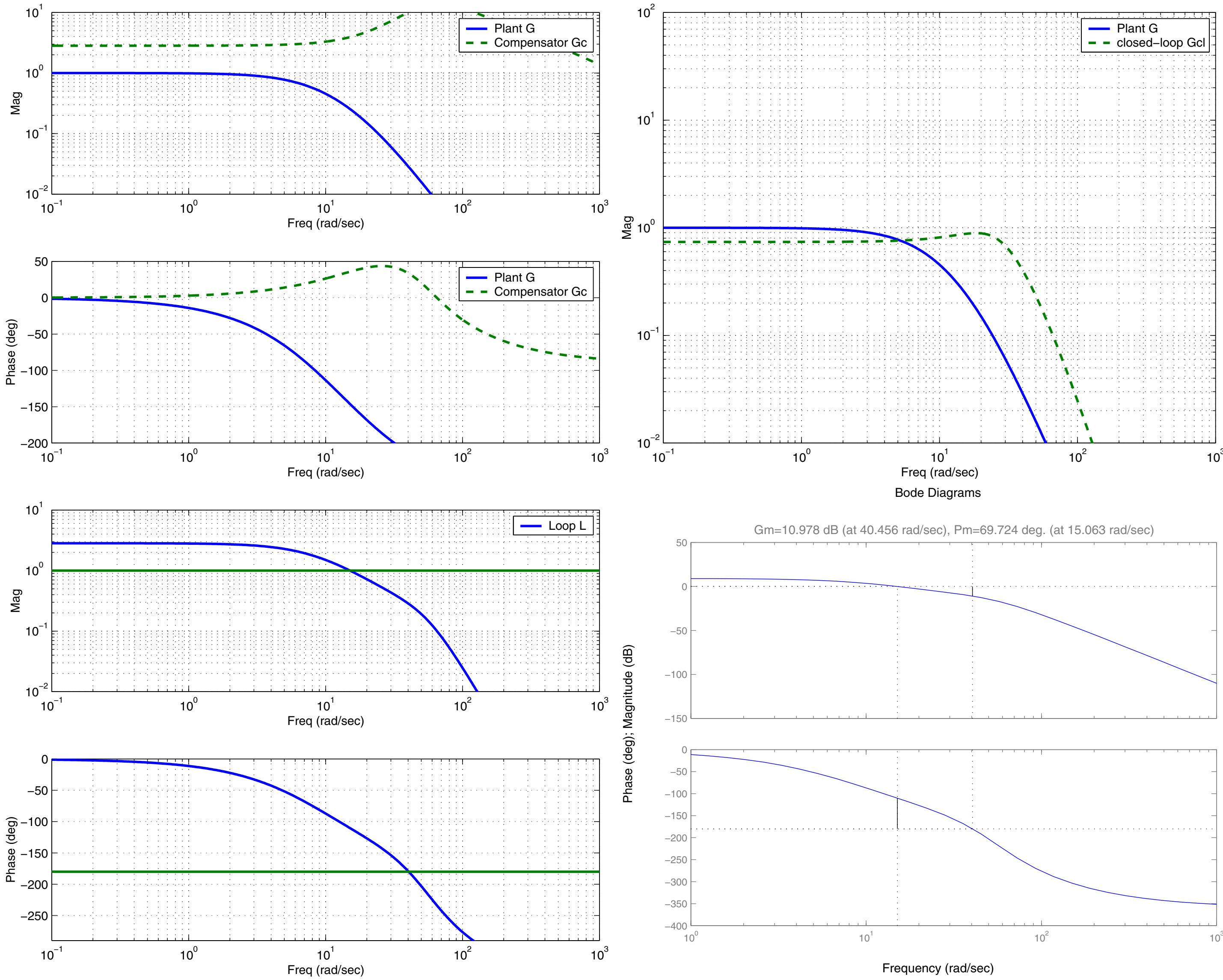
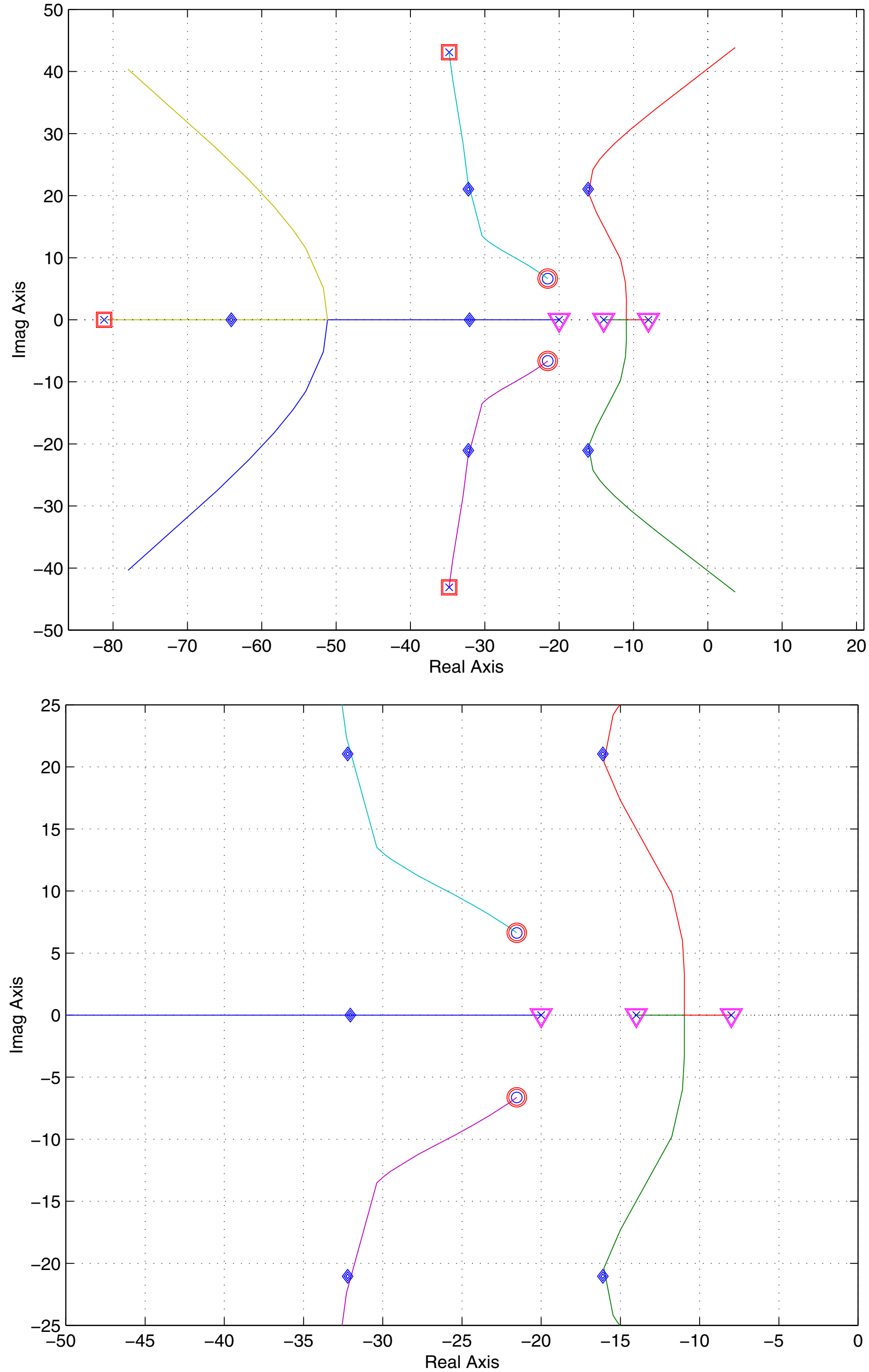


Figure 6: Example #1: $G(s) = \frac{8 \cdot 14 \cdot 20}{(s+8)(s+14)(s+20)}$



◇ – closed-loop poles, ▽ – open-loop poles, □ – Compensator poles, ○ – Compensator zeros

- Two compensator zeros at -21.54±6.63j draw the two lower frequency plant poles further into the LHP.
- Compensator poles are at much higher frequency.
- Looks like a lead compensator.

Figure 7: Example #2: $G(s) = \frac{0.94}{s^2 - 0.0297}$

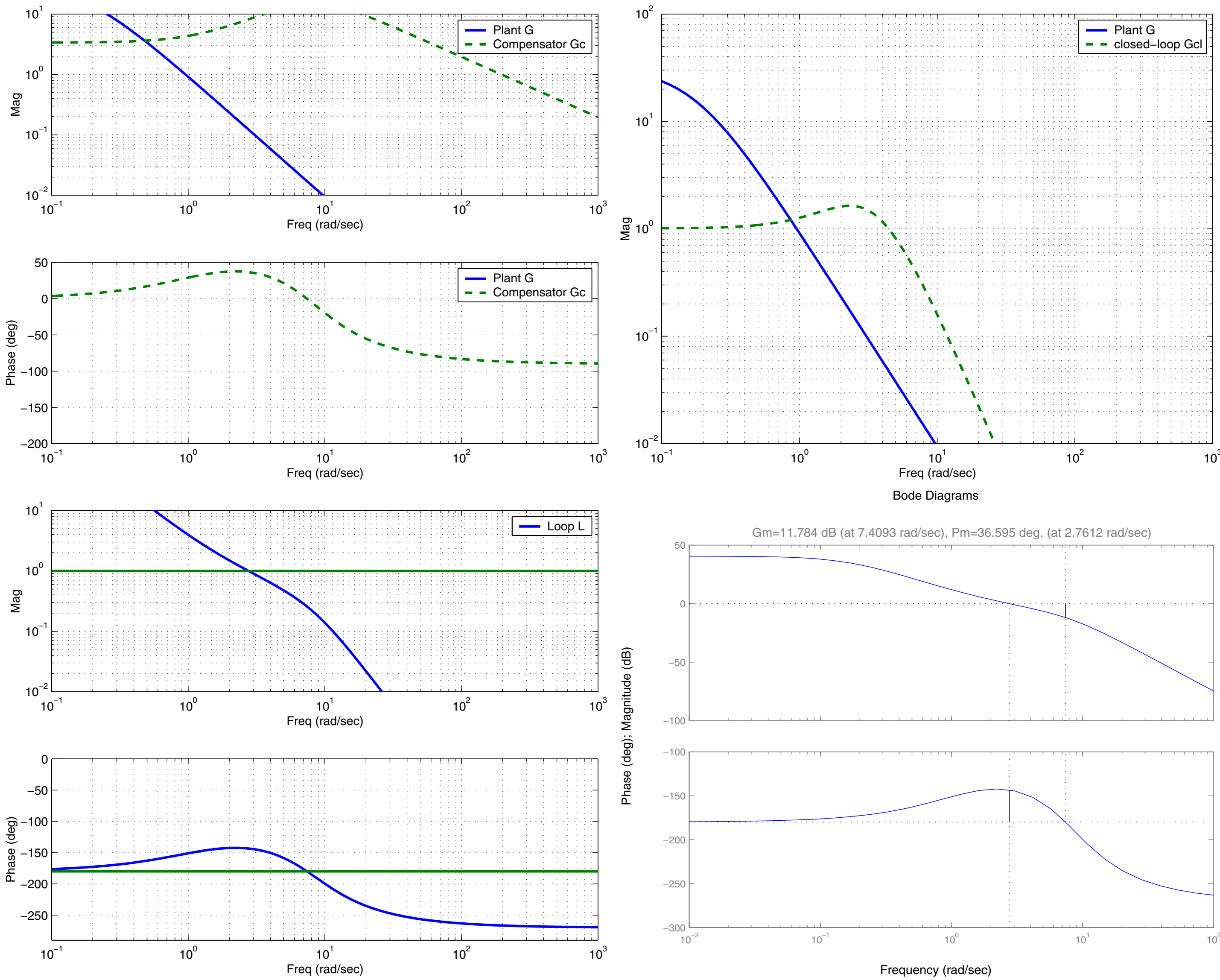
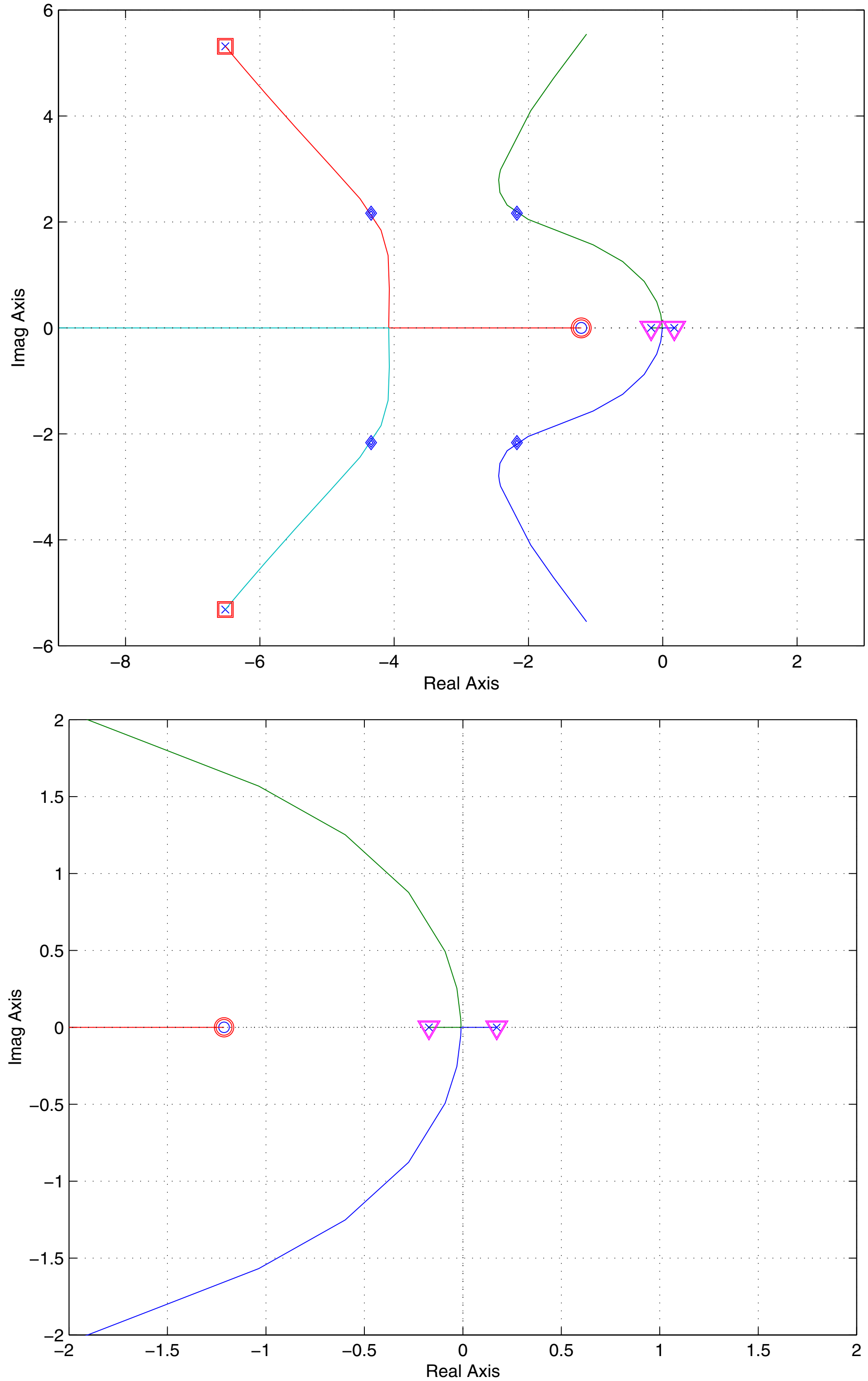


Figure 8: Example #2: $G(s) = \frac{0.94}{s^2 - 0.0297}$



◇ – closed-loop poles, ▽ – open-loop poles, □ – Compensator poles, ○ – Compensator zeros

- Compensator zero at -1.21 draws the two lower frequency plant poles further into the LHP.
- Compensator poles are at much higher frequency.
- Looks like a lead compensator.

Figure 9: Example #3: $G(s) = \frac{8 \cdot 14 \cdot 20}{(s-8)(s-14)(s-20)}$

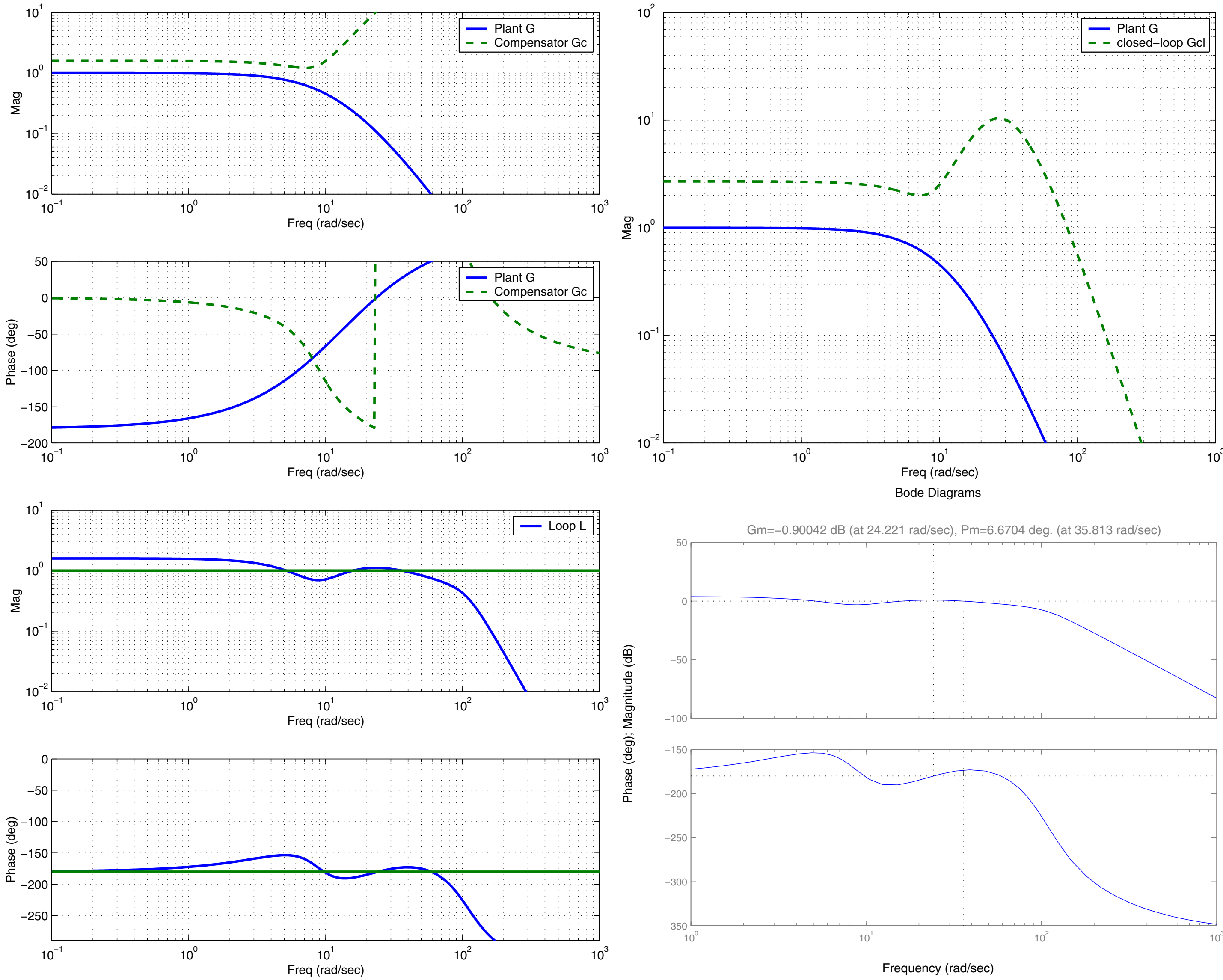
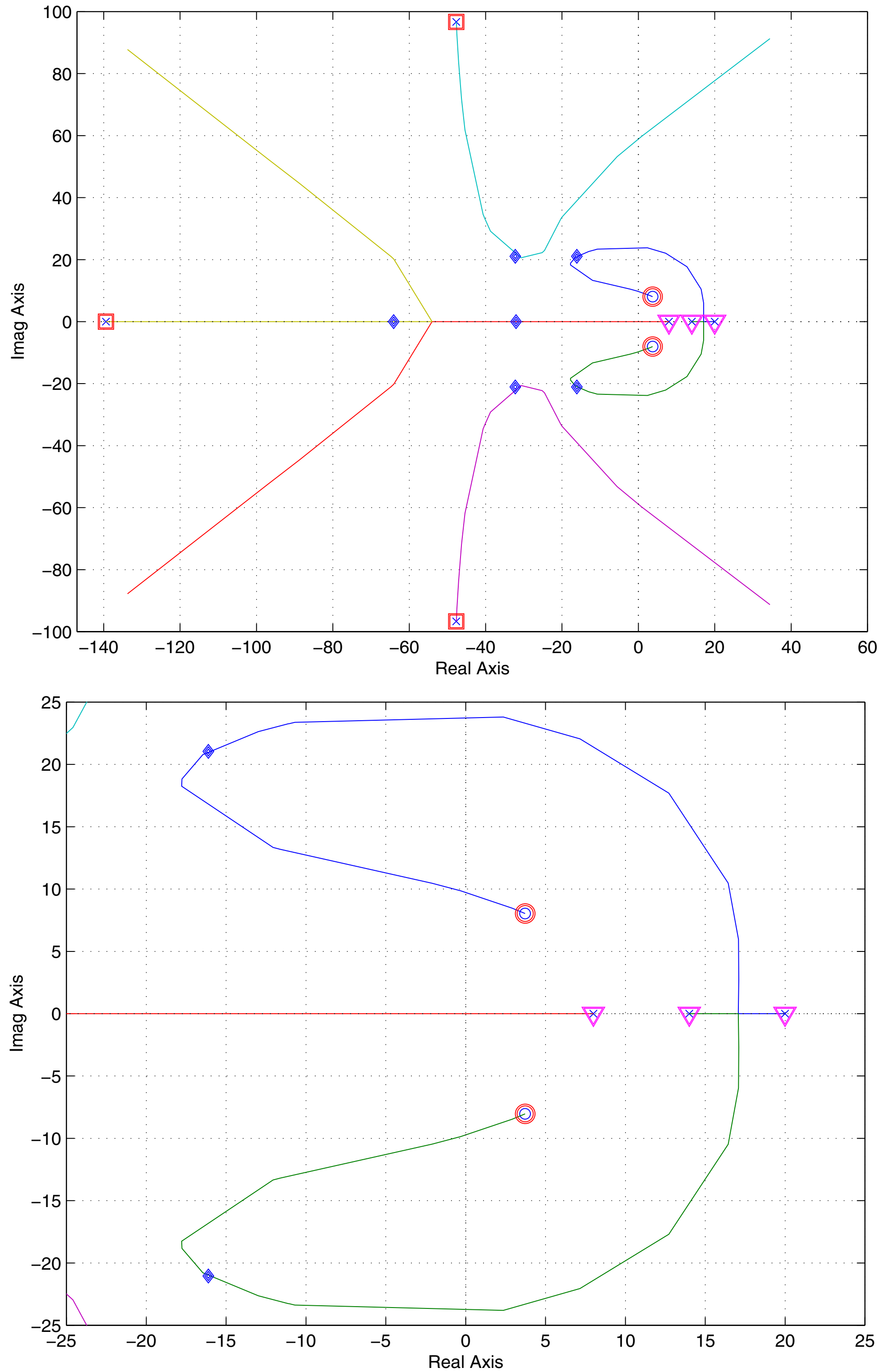


Figure 10: Example #3: $G(s) = \frac{8 \cdot 14 \cdot 20}{(s-8)(s-14)(s-20)}$



◇ – closed-loop poles, ▽ – open-loop poles, □ – Compensator poles, ○ – Compensator zeros

- Compensator zeros at $3.72 \pm 8.03j$ draw the two higher frequency plant poles further into the LHP. Lowest frequency one heads into the LHP on its own.
- Compensator poles are at much higher frequency.
- Note sure what this looks like.

Figure 11: Example #4: $G(s) = \frac{(s-1)}{(s+1)(s-3)}$

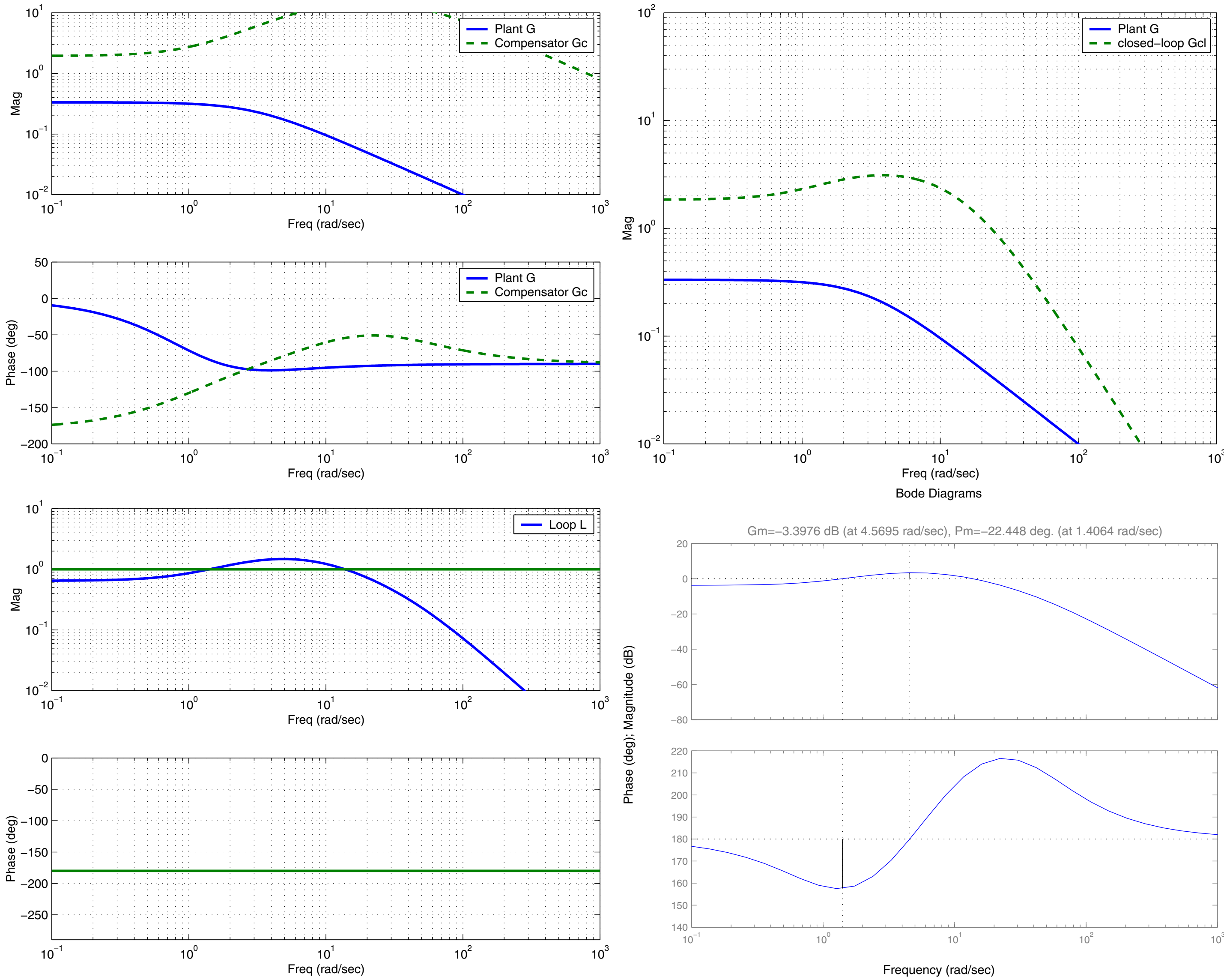
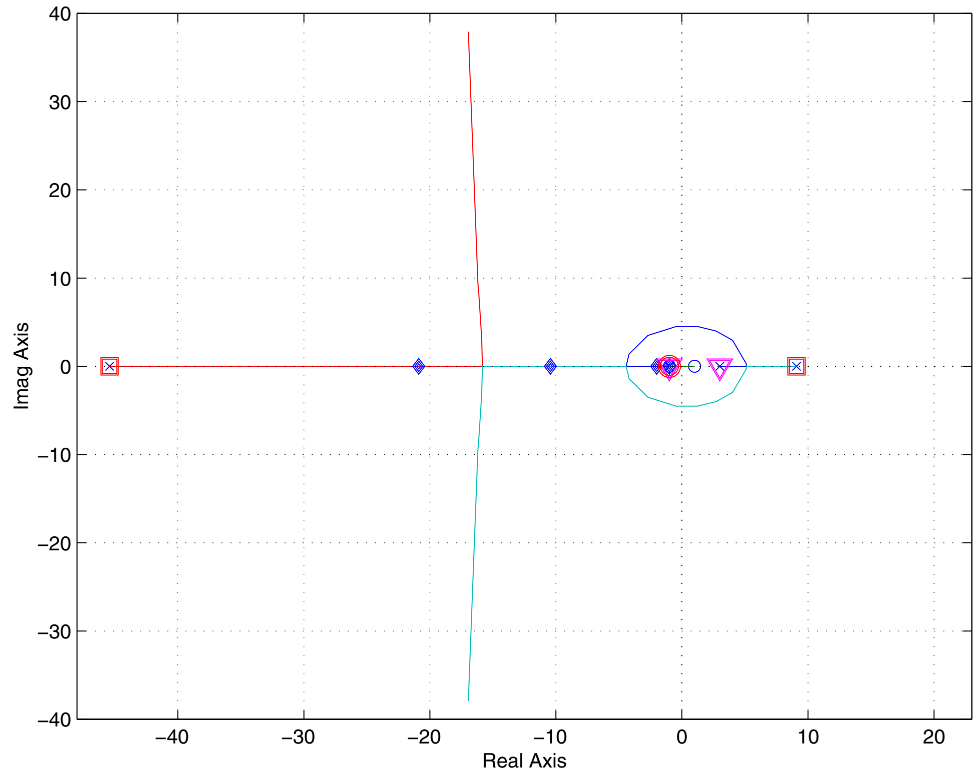
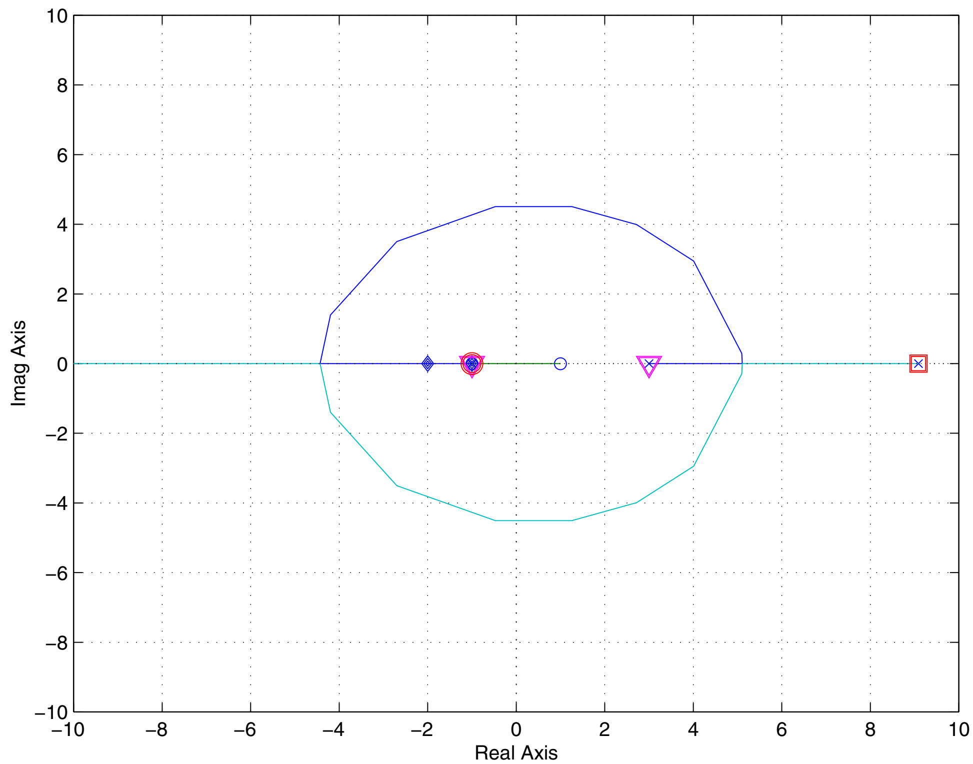


Figure 12: Example #4: $G(s) = \frac{(s-1)}{(s+1)(s-3)}$





◇ – closed-loop poles, ▽ – open-loop poles, □ – Compensator poles, ○ – Compensator zeros

- Compensator zero at -1 cancels the plant pole. Note the very unstable compensator pole at $s = 9$!!
  - Needed to get the RHP plant pole to branch off the real line and head into the LHP.
- Other compensator pole is at much higher frequency.
- Note sure what this looks like.

- Separation principle gives a very powerful and simple way to develop a dynamic output feedback controller

- Note that the designer now focuses on selecting the appropriate regulator and estimator pole locations. Once those are set, the closed-loop response is specified.
  - Can almost consider the compensator to be a by-product.

- These examples show that the design process is extremely simple.