Fall 2001

Topic #5

16.31 Feedback Control

Stability in the Frequency Domain

- Nyquist Stability Theorem
- Examples
- Appendix (details)

- Remember that this is the basis of future robustness tests.

STABILITY IN THE FREQ. DOMAIN

- LOOKING FOR TESTS ON THE LOOP TRANSFER FUNCTION $G_c(s)\,G(s)$ THAT WE CAN PERFORM TO ESTABLISH STABILITY OF THE CLOSED-LOOP SYSTEM $G_{CL}(s) = \dfrac{G_c G}{1 + G_c G}$ $\boxed{\text{LOOP } L = G_c G}$

  - ROOT LOCUS ONE EASY WAY (POLES AND ZEROS OF PRODUCT DEFINE LOCUS OF CLOSED-LOOP ROOTS ...)

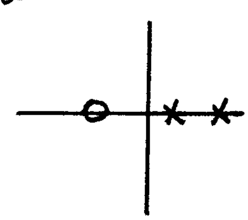
  - HOW DO THIS IN FREQUENCY DOMAIN (BODE PLOT)? $\Rightarrow$ WHAT IS OUR SIMPLE EQUIVALENT TEST TO "DOES THE LOCUS GO INTO THE RHP"?

- INTUITION: ALL POINTS ON THE ROOT LOCUS HAVE THE PROPERTIES i) $\angle G_cG(s) = \pm 180°$

  ii) $|G_cG(s)| = 1$

$\Rightarrow$ AT THE POINT OF NEUTRAL STABILITY (IMAG AXIS CROSSING), WE KNOW THIS HOLDS FOR $s = j\omega$

MUST HAVE $\Rightarrow$ $|G_cG(j\omega)| = 1$; $\angle G_cG(j\omega) = 180°$



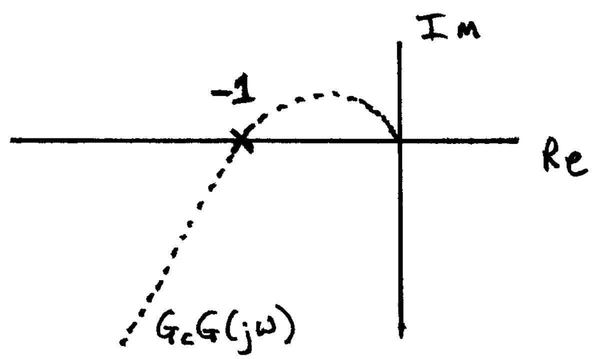
- CONDITIONS THAT THE BODE PLOT MUST SATISFY FOR NEUTRAL STABILITY.

- SO, TYPICALLY, WE WOULD EXPECT TO SEE $|G_cG(j\omega)| < 1$ AT THE FREQUENCIES THAT $\angle G_cG(j\omega) = 180°$ $\Rightarrow$ STABLE

- TO BE MORE PRECISE ABOUT THIS STATEMENT OF STABILITY WE NOTE THAT

$$\left.\begin{array}{l} |G_cG(j\omega)| = 1 \\ \angle G_cG(j\omega) = 180° \end{array}\right\} \Leftrightarrow \quad G_cG(j\omega) = -1 + 0j$$

$\therefore$ THE TEST FOR <u>NEUTRAL</u> <u>STABILITY</u> (AS GIVEN) IS THAT (AT SOME FREQUENCY), THE PLOT OF $G_cG(j\omega)$ IN THE COMPLEX PLANE PASS THROUGH THE "-1 POINT" (CRITICAL POINT)



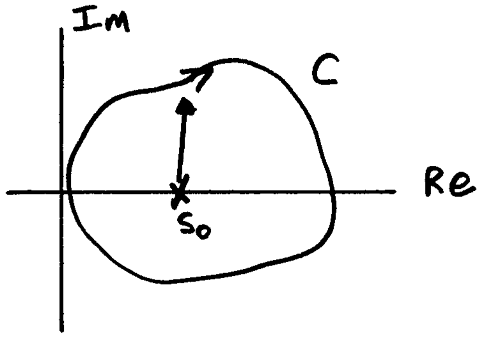
NEUTRALLY STABLE.

- NICE OBSERVATION, BUT WE STILL NEED TO BE CAREFUL – THE FIRST STATEMENT ON THIS PAGE ASSUMES THAT ① INCREASING GAIN LEADS TO INSTABILITY AND ② $|G_cG(j\omega)| = 1$ AT ONE FREQ. GOOD ASSUMPTIONS BUT <u>NOT</u> ALWAYS TRUE.

NYQUIST STABILITY THEOREM

- TO BE MORE PRECISE, WE ARE NOT GOING TO JUST WORRY ABOUT $L(s) = G_c(s)G_p(s)$ PASSING THROUGH -1, BUT HOW MANY TIMES IT ENCIRCLES IT.
  - ALSO TAKE INTO ACCOUNT THE STABILITY OF $L(s)$.

"AN ENCIRCLEMENT"



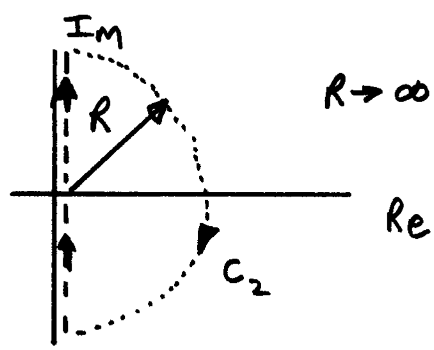
- ACCUMULATION OF 360° OF PHASE BY A VECTOR (TAIL AT s_0) AS THE TIP TRAVERSES THE CONTOUR "C"

$\Rightarrow$ C ENCIRCLES s_0 (CW)

- WE ARE INTERESTED IN THE PLOT OF $L(s)$ FOR VERY SPECIFIC VALUES OF "s".

NYQUIST PATH

(DASHED LINE)



ASSUME $L(s)$ HAS NO IMAGINARY AXIS POLES

$\Rightarrow$ NYQUIST DIAGRAM: PLOT OF $L(s)$ AS "s" MOVES AROUND C_2

- **STEP 1)**: - DETERMINE NYQUIST PATH (IMAG AXIS)
  2) - DRAW NYQUIST PLOT/DIAGRAM
  3) - COUNT # ENCIRCLEMENTS OF -1

- WHY DO WE CARE ABOUT THE # OF ENCIRCLEMENTS?
  - TURNS OUT (SEE APPENDIX) THAT IF $L(s)$ HAS POLES IN RHP, THEN THE NYQUIST PLOT MUST ENCIRCLE THE -1 POINT FOR THE CLOSED-LOOP SYSTEM TO BE STABLE.

  $\Rightarrow$ OUR JOB IS TO MAKE SURE THAT WE HAVE ENOUGH ENCIRCLEMENTS.

- HOW MANY DO WE NEED? → **NYQUIST THEOREM**

LET P = # POLES OF $L(s)$ IN THE RHP

Z = # CLOSED-LOOP POLES IN THE RHP

N = # CLOCKWISE ENCIRCLEMENTS OF NYQUIST DIAGRAM ABOUT -1

- CAN SHOW THAT $Z = N + P$
- FOR STABILITY WE NEED $Z = 0$ {NO CLP RHP POLES}

$$\Rightarrow \quad N = -P$$

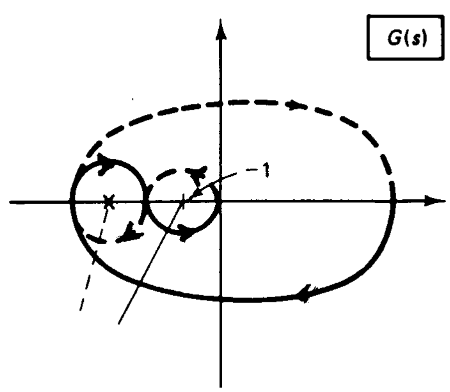
$\boxed{P>0, \therefore \text{ EXPECT CCW ENCIRCLEMENTS}}$

- SEE APPENDIX.

- WE WILL NOT SPEND TOO MUCH TIME DRAWING NYQUIST PLOTS → WILL SEE THAT WE CAN EXTRACT SOME RULES-OF-THUMB, AND USE THOSE IN THE BODE PLOTS

  ⇒ BUT IF THE STABILITY IN THE BODE PLOT IS UNCLEAR, GO BACK TO THE NYQUIST TEST!

- THE WHOLE ISSUE WITH THE NYQUIST TEST BOILS DOWN TO MAKING AN ACCURATE PLOT + BEING CAREFUL HOW TO COUNT N



GOOD APPROACH IS TO FIND THE # OF CROSSINGS OF POINT(S.)

- DRAW A LINE FROM S.
- COUNT THE # OF TIMES THAT THE LINE AND DIAGRAM CROSS.

⇒ N = # CW CROSSINGS − # CCW CROSSINGS.

- **BOTTOM LINE**: THE # OF ENCIRCLEMENTS IS CRUCIAL. FOR A STABLE SYSTEM AND STABLE CONTROLLER, WOULD EXPECT NO ENCIRCLEMENTS.

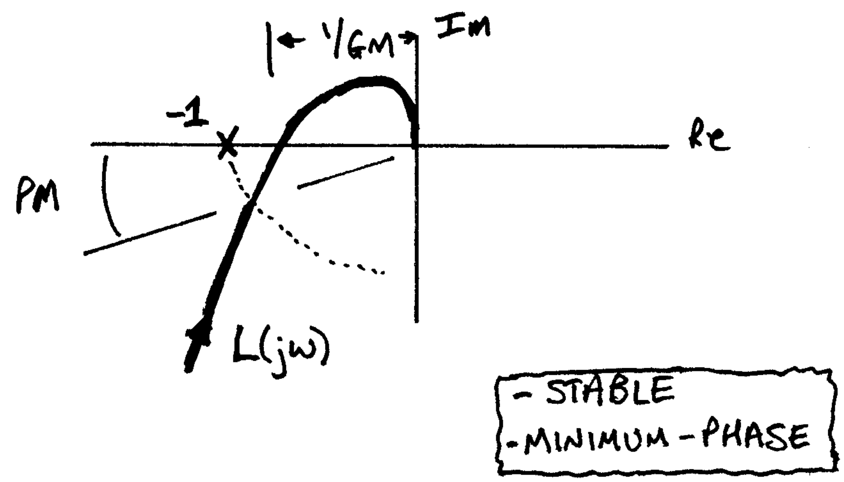
  $\Rightarrow$ HOWEVER WE MIGHT FIND THAT THE PLOT OF $L(s)$ ALONG THE NYQUIST CONTOUR GETS VERY CLOSE TO -1.

  $\Rightarrow$ STABLE, BUT MARGINALLY.

  - HOW QUANTIFY THIS?

- VERY GENERIC PLOT OF $L(j\omega)$ FOR $\omega \geq 0$

  $\Rightarrow$ NO ENCIRC OF -1 (L STABLE)



- NOTE THAT IF WE WERE TO CHANGE THE $|L|$ OR $\angle L$ (EASILY DONE THROUGH $G_c(s)$), THEN WE COULD CHANGE THE # OF ENCIRC.

  $\Rightarrow$ ACTUALLY BECOMES HOW WE DO THE SYNTHESIS!

- <u>GAIN MARGIN (GM)</u> [IN dB]
  - FACTOR BY WHICH THE GAIN IS LESS THAN 1 AT THE FREQUENCIES THAT $\angle L = 180° = \omega_\pi$

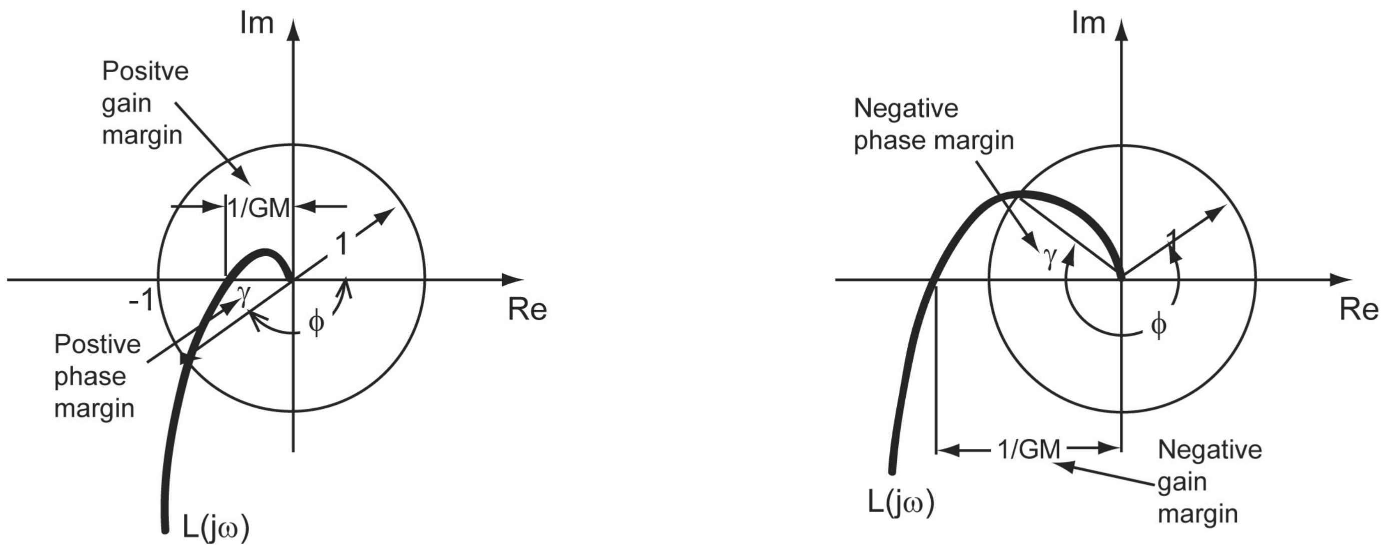
$$GM[dB] = -20 \log |G(j\omega_\pi)|$$

- <u>PHASE MARGIN</u>
  - ANGLE THAT NYQUIST PLOT MUST BE ROTATED SO THAT IT INTERSECTS THE -1 POINT.
  - LET ω_c = FREQ THAT $|L(j\omega)| = 1$

$$\phi = \angle L(j\omega_c)$$

$$PM = 180° + \phi$$

- FOR TYPICAL SYSTEM (STABLE, MINIMUM PHASE $L(s)$) WE NEED BOTH $GM > 0$ AND $PM > 0$

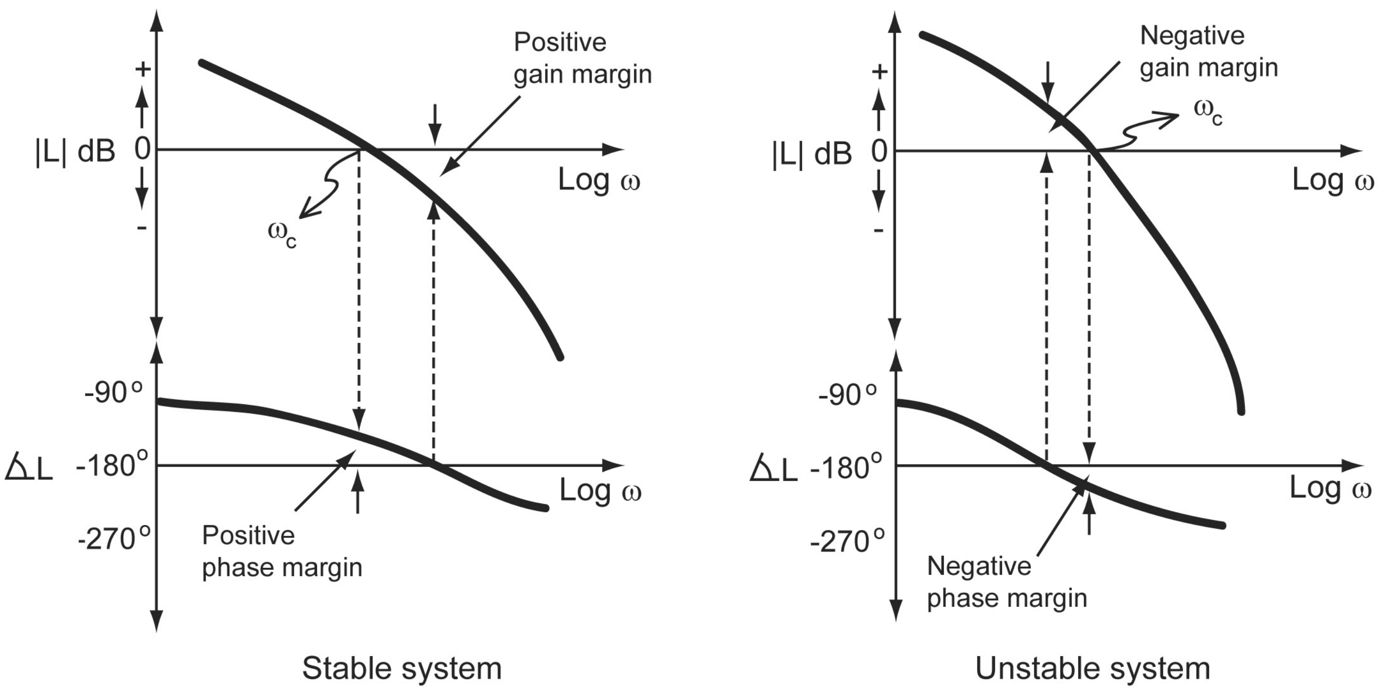


Stable system

Unstable system

- POLAR PLOT FOR "TYPICAL" (STABLE, MIN PHASE) $L(j\omega)$, $\omega \geq 0$

- WHILE THE NYQUIST STABILITY THEOREM GIVES US THE STABILITY CONDITIONS, THEY ARE OFTEN (NOT ALWAYS) EASIER TO FIND ON THE BODE PLOT.



⇒ CAN PREDICT CLOSED-LOOP STABILITY BY LOOKING AT THE GAIN AND PHASE MARGIN.

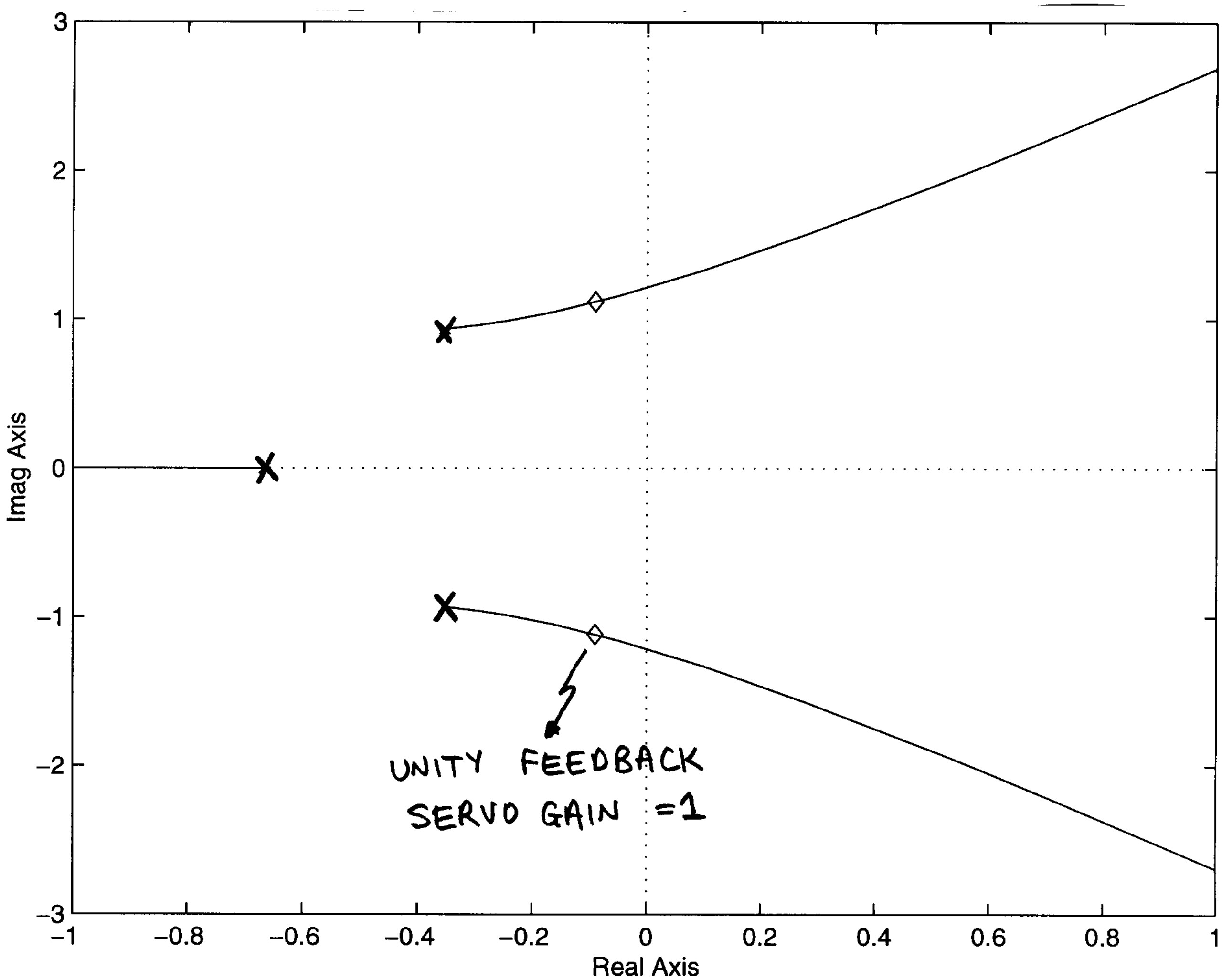
- WE SAY "OFTEN" BECAUSE, IF $L(s)$ UNSTABLE, WE KNOW THAT AN ENCIRCLEMENT OF -1 IS REQUIRED → TOUGH TO SEE THIS IN BODE PLOT.

  - BODE PLOT DRAWING ALSO COMPLICATED FOR NMP SYSTEMS, BUT GM/PM USAGE THE SAME.

⇒ OFTEN BETTER TO USE NYQUIST FOR THESE SYSTEMS, JUST TO BE SAFE.

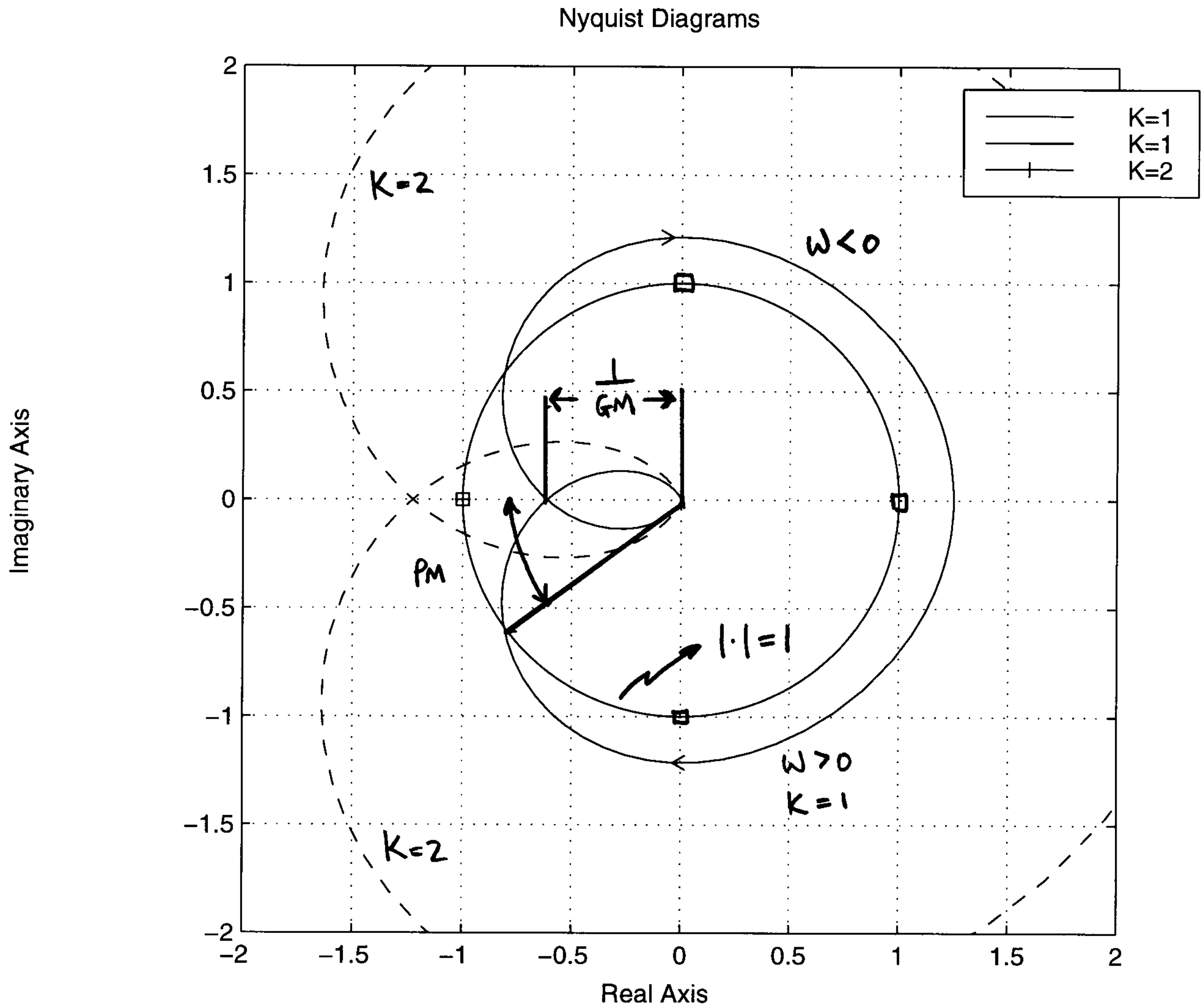
EXAMPLE # 1 - STABLE, MIN PHASE

$$G(s) = \frac{1.25}{(1.5s+1)(s^2+0.707s+1)}$$



- ROOT LOCUS PREDICTS STABLE FOR K= 1, BUT EXPECT PROBLEMS AT HIGHER GAIN.

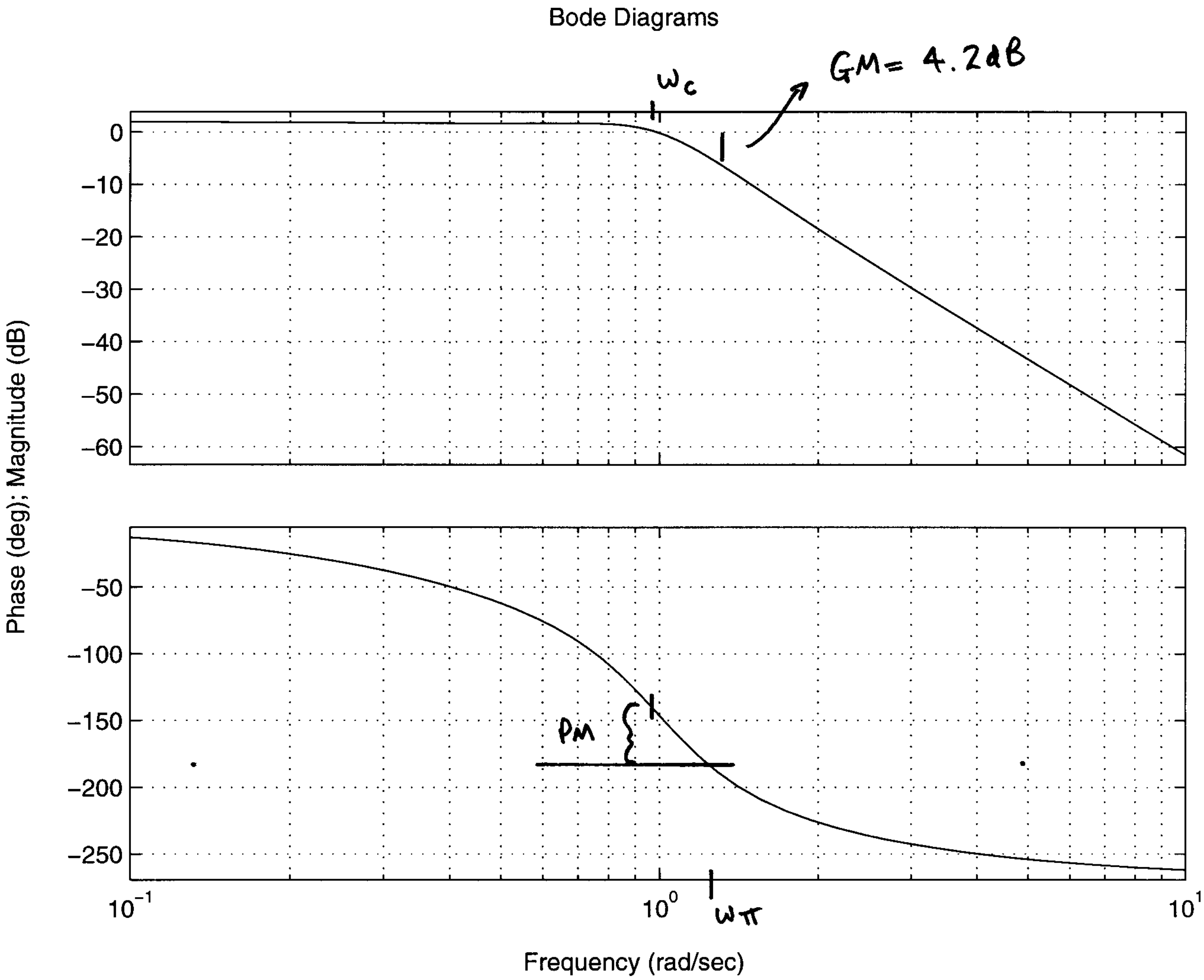
EX #1



→ STABLE K=1, UNSTABLE K=2

∴ CONSISTENT WITH ROOT LOCUS

EX #1



GM / PM EASY TO DETERMINE
FOR K=1 AND K=2.

EXAMPLE: $G(s) = \dfrac{2s+1}{(s+0.3)(s-2)}$ $\longrightarrow s^2-1.7s-0.6$

- STABILITY WITH $G_c = K$?

- ROOT LOCUS – STARTS IN RHP, BUT THERE IS A ZERO IN THE LHP, SO CLP MIGRATE THERE.

  ⇒ THERE IS A $K\ (=-0.85)$ FOR WHICH THE POLES CROSS THE IMAGINARY AXIS.

CLP SOLVE $1 + KG = 0 \Rightarrow (s^2-1.7s-0.6) + K(2s+1) = 0$

AT THE IMAG AXIS CROSSING, THIS IS TRUE FOR $s = j\omega$

$$\Rightarrow \quad (j\omega)^2 + (j\omega)(2K-1.7) + (K-0.6) = 0 + 0j$$

$$\therefore \; 2K-1.7 = 0, \quad \text{OR} \quad K = +0.85$$

☺

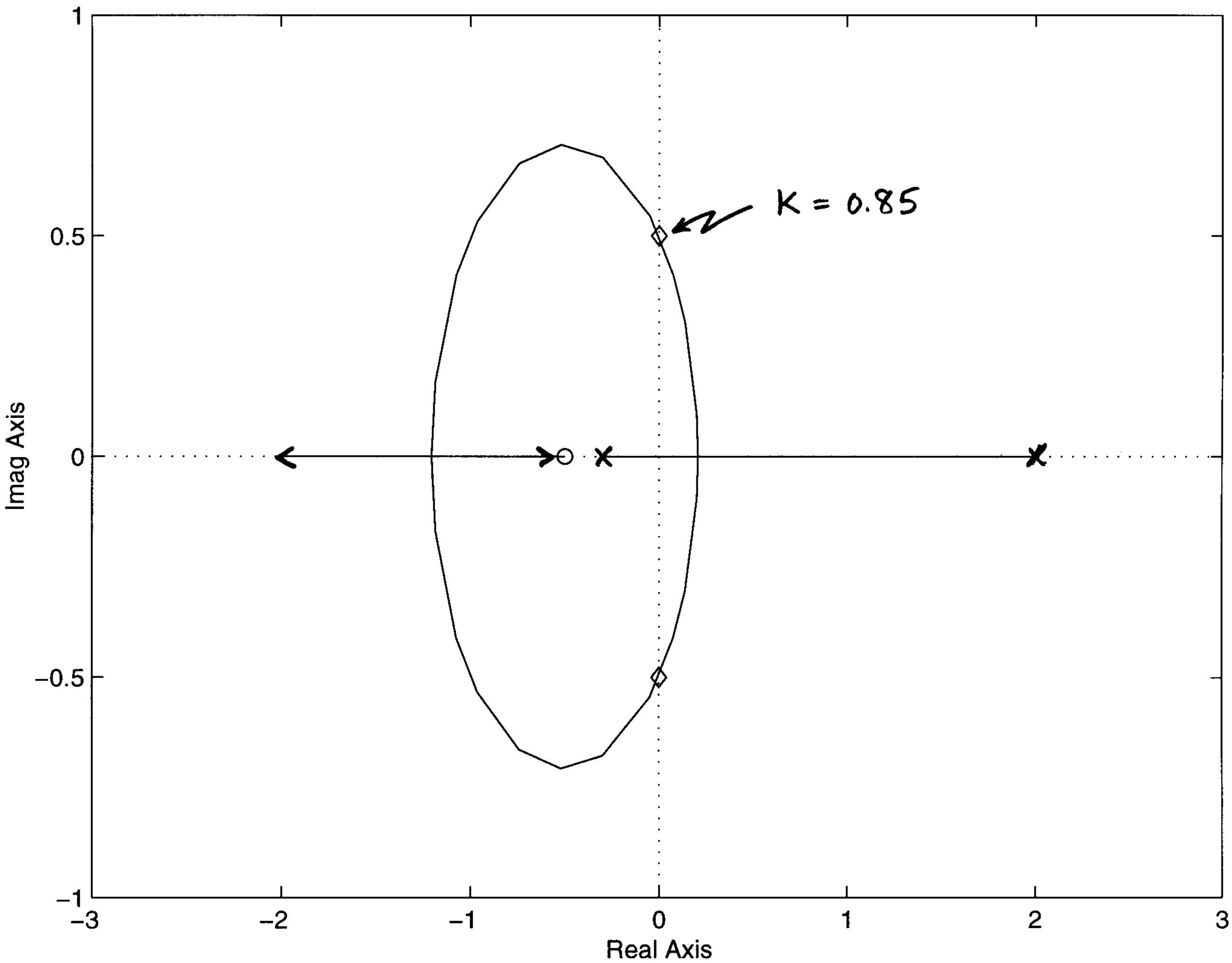
- SINCE $G(s)$ HAS A RHP POLE, $L(s)$ MUST HAVE ONE

  ⇒ EXPECT THE NYQUIST PLOT TO ENCIRCLE -1 ONCE CCW $(N = -P,\ P = 1)$

  - LOOK AT PLOT:
    - WITH $K = 1$ WE HAVE CORRECT # ENCIRCLEMENTS
    - WITH $K \le 0.85$ WE WOULD EXPECT TO BE IN TROUBLE.

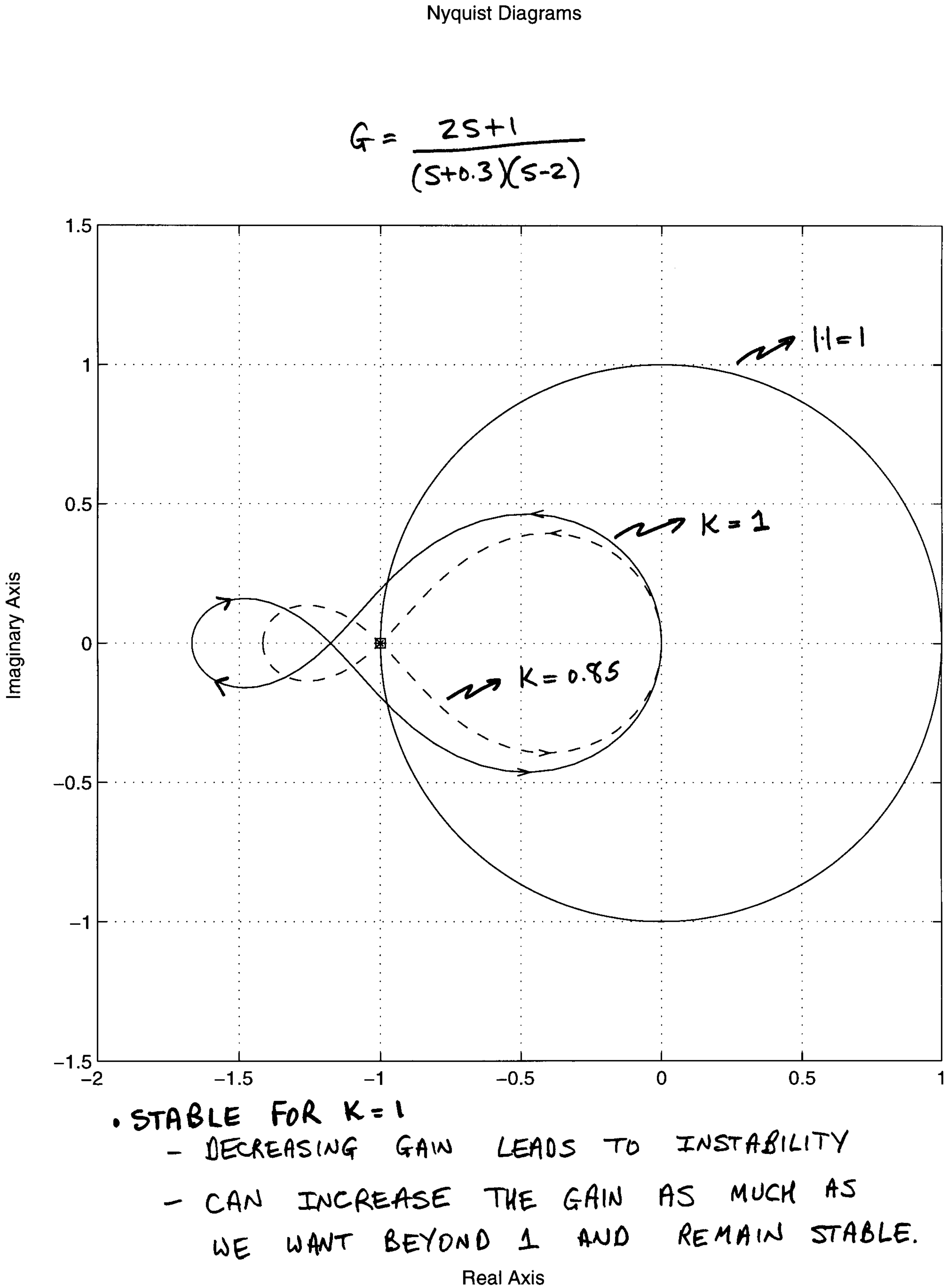
- NOT AS OBVIOUS WHAT IS GOING ON IN THE BODE PLOT $|L| > 1$ WHEN $\angle L = 180°$ "CONDITIONALLY STABLE"

$$G = \frac{2s+1}{(s+0.3)(s-2)}$$

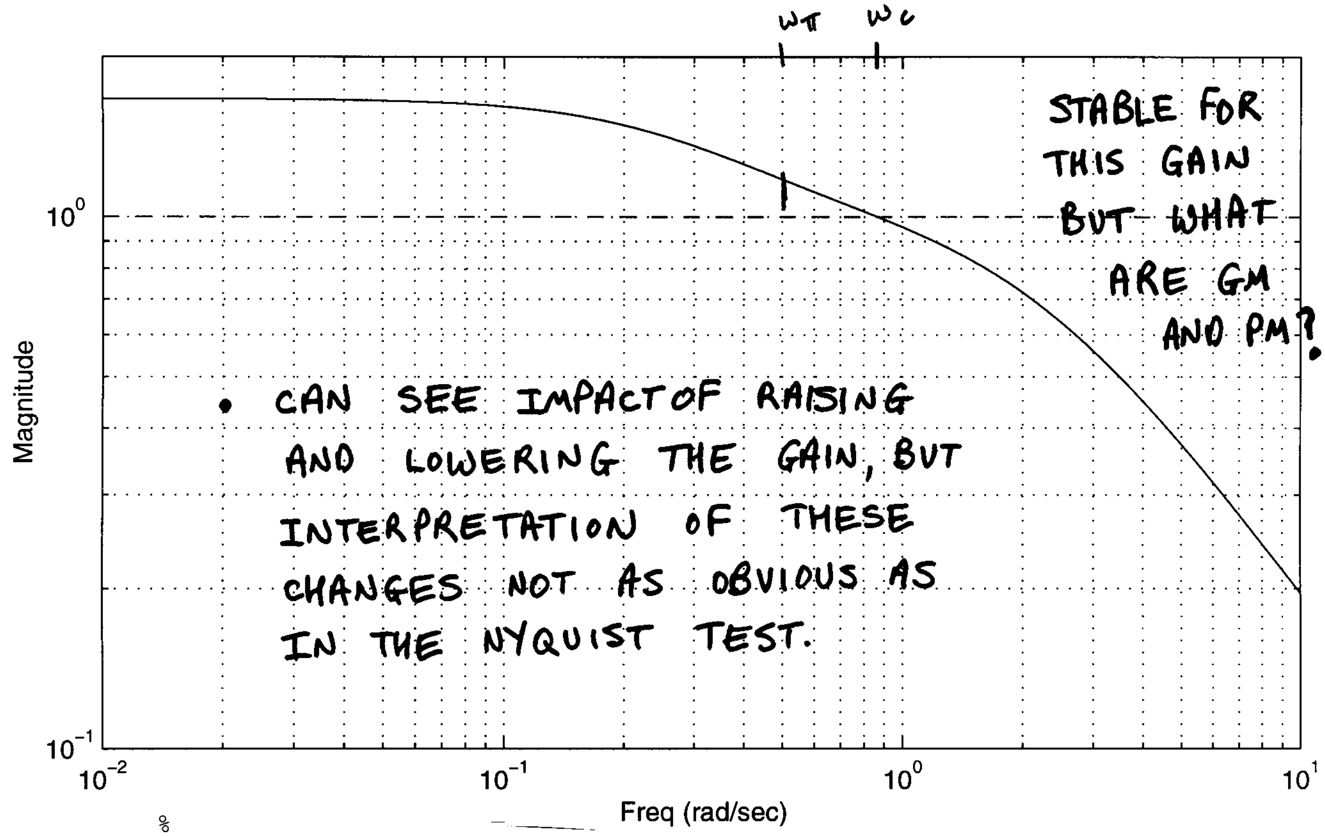


- CALLED "CONDITIONAL STABLE" SINCE WE REQUIRE A GAIN OF AT LEAST 0.85 TO STABILIZE.

$$G = \frac{2s+1}{(s+0.3)(s-2)}$$



- STABLE FOR K = 1
  - DECREASING GAIN LEADS TO INSTABILITY
  - CAN INCREASE THE GAIN AS MUCH AS WE WANT BEYOND 1 AND REMAIN STABLE.



STABLE FOR THIS GAIN BUT WHAT ARE GM AND PM?

- CAN SEE IMPACT OF RAISING AND LOWERING THE GAIN, BUT INTERPRETATION OF THESE CHANGES NOT AS OBVIOUS AS IN THE NYQUIST TEST.

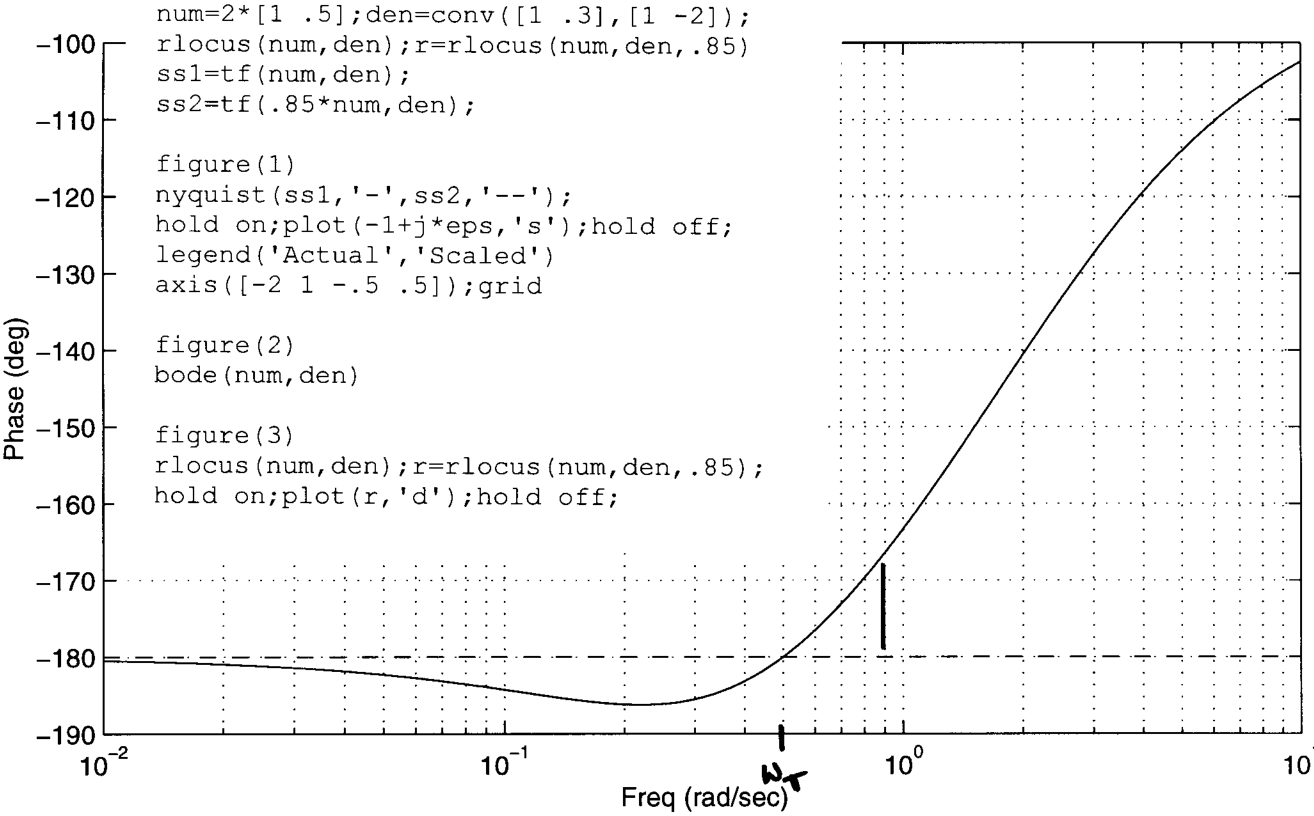
```
%
% simple example of an unstable system
% to compare the various stability tests
% imag axis crossing for k=0.85
%
num=2*[1 .5];den=conv([1 .3],[1 -2]);
rlocus(num,den);r=rlocus(num,den,.85)
ss1=tf(num,den);
ss2=tf(.85*num,den);

figure(1)
nyquist(ss1,'-',ss2,'--');
hold on;plot(-1+j*eps,'s');hold off;
legend('Actual','Scaled')
axis([-2 1 -.5 .5]);grid

figure(2)
bode(num,den)

figure(3)
rlocus(num,den);r=rlocus(num,den,.85);
hold on;plot(r,'d');hold off;
```

$$G = \frac{2s+1}{(s+0.3)(s-2)}$$



SUMMARY

- FOR MOST SYSTEMS WE DESIGN THE COMPENSATOR TO GET $GM > 0$, $PM > 0$.

- MUST BE CAREFUL WITH INTERPRETATION OF BODE PLOTS FOR UNSTABLE (OPEN-LOOP) SYSTEMS

  $\Rightarrow$ USE A VARIETY OF TESTS TO DOUBLE CHECK.

- NORMALLY SHOOT FOR $\begin{cases} PM \sim 30\text{-}60^\circ & \text{AND} \\ GM \sim 6\ dB \end{cases}$

NYQUIST STABILITY THEOREM

N # CLOCKWISE ENCIRCLEMENTS OF NYQUIST DIAGRAM ABOUT -1

P # POLES $G_c(s)G_p(s)$ IN THE RHP

Z # CLOSED-LOOP POLES IN THE RHP.

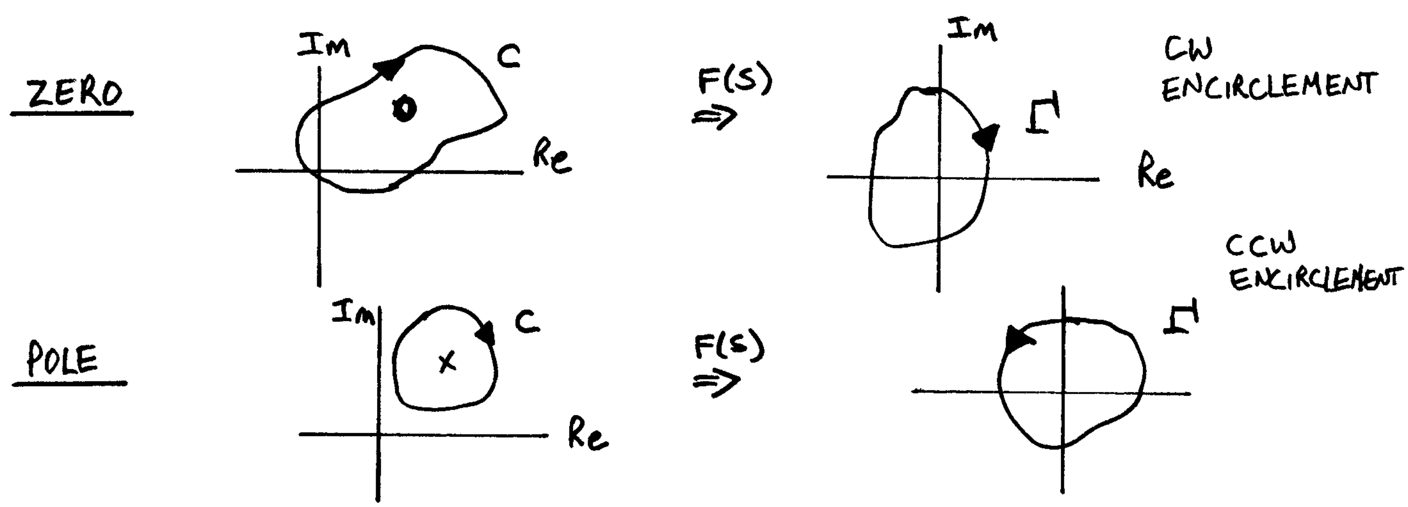
CAN SHOW THAT $Z = N + P$

CLEARLY, FOR STABILITY, WE NEED $Z = 0 \Rightarrow N = -P$

OUTLINE OF PROOF:

- PROOF BASED ON UNDERSTANDING OF HOW FUNCTIONS ($F(s)$) MAP CONTOURS IN THE S-PLANE.

  $\Rightarrow$ MAP OF $F(s)$ WILL ONLY ENCIRCLE THE ORIGIN IF THE CONTOUR IN THE S-PLANE CONTAINS A POLE OR ZERO OF $F(s)$.



$\Rightarrow$ CAN ALSO HAVE MIXTURES OF POLES (n_p) + ZEROES (n_z) IN "C", Γ ENCIRCLES THE ORIGIN $n_z - n_p$

HOW APPLY THIS?

- USE $F(s) = 1 + G_c(s)\,G_p(s)$
- USE NYQUIST PATH FOR "C" (*)
- PLOT $F(s)$ ALONG "C" + COUNT # ENCIRCLEMENTS AROUND THE ORIGIN (N)

$$\Rightarrow \quad N = Z - P$$

ZEROES OF F(S)

POLES OF F(S)

NOTE:

Z = # ZEROES OF F(S) = CLOSED-LOOP POLES OF SYSTEM

P = # POLES OF F(S) IN "C" = UNSTABLE POLES OF $G_c(s)\,G_p(s)$

- USUALLY PLOT $G_c(s)\,G_p(s)$ AND LOOK AT ENCIRCLEMENTS OF $s = -1$ (SAME THING)

- NOTE: $F(s) = 1 + G_c(s)\,G(s)$

  $\therefore$ ZEROS OF F(S) ARE THE CLP POLES

  POLES OF F(S) ARE THE LOOP POLES

  $\Rightarrow$ ONLY CARE ABOUT THE CLP AND LOOP POLES IN THE RHP.