Topic #6

16.31 Feedback Control

Control Design using Bode Plots

- Performance Issues
- Synthesis
- Lead/Lag examples
Bode's Gain-Phase Relationship

- Synthesis would be hard if we had to deal with both magnitude and phase plots, but ...

- **Theorem**: For any stable, minimum phase system, \( \angle G(j\omega) \) is uniquely related to \( |G(j\omega)| \).

- **Relationship**: On log-log plot, if slope of magnitude curve is constant over a decade of frequency, with slope \( n \), then

  \[ \angle G(j\omega) \approx 90^\circ \cdot n \]

  **Very Important**

- So, in the cross-over frequency range \( (|G_c G_p| \approx 1) \), if the slope is:

  \[ S^0 \quad \text{no cross-over} \]
  \[ S^{-1} \approx 90^\circ \]
  \[ S^{-2} \approx 180^\circ \quad \text{too much phase as PM=0} \]

\[ \Rightarrow \text{Select } G_c(s) \text{ so that LTF crosses over with a slope of -1} \]
OTHER PERFORMANCE ISSUES

- **Step Response Error** \( e_{ss} = \frac{1}{1 + G_c(0) G_p(0)} \)

  **Good News:** We can find \( G_c G_p(0) \) directly from the low freq bode plot (Type 0)

- For Type I systems, the low freq asymptote is infinite (slope -1)
  
  \( \Rightarrow \) step error of zero
  
  \( \Rightarrow \) velocity error (ramp input) \( e_{ss} = \frac{1}{K_v} \)

  \( \Rightarrow K_v \) is the gain of the curve at low freq

  \[ K_v = \lim_{s \to 0} s \cdot G_c(s) G_p(s). \]

- For Type I systems \( G_c G_p(s) \approx \frac{K_v}{s} \)
  
  At low frequency

  \( \Rightarrow \) can find \( K_v \) by extending the low frequency asymptote (if necessary)

  To \( \omega = 1 \) and finding \( |G_c G_p| \) at \( \omega = 1 \)

  \[ K_v = \omega \cdot |G_c G_p| \]
**PERFORMANCE**

- **How much phase margin?**
  
  The response of a 2nd order system gives:

  1) **Damping ratio of CLP poles** \( \xi = \frac{PM}{100} \), \( PM < 70^\circ \)
  
  2) **CLP resonant peak** \( M_r = \frac{1}{2 \sin\left(\frac{PM}{2}\right)} \)
  
  3) \( \omega_{BW} \approx 1.4 \omega_c \)

  CLP bandwidth \( \approx 1.4 \times \) cross over frequency.

- **Peak of TF usually close to \( \omega_c \)**

  \( \Rightarrow \) specify (ultimately) \( \omega_c \) and \( PM \)
**FREQUENCY RESPONSE DESIGN**

- Looked at building block \( G_c(s) = \frac{k_c(s+z)}{(s+p)} \)
  - Root locus applications

- Frequency response characteristics?
  - How use \( G_c(s) \) to modify the loop transfer function (LTF) \( G_c(s)G_p(s) \)
  - To get desired - bandwidth
  - Phase margin.
  - Error constants.

1. **Lead** \( |z| < |p| \)
   - Zero at lower frequency than pole
     - Gain increases with frequency
     - Phase positive (i.e. adds phase lead)

![Graph](image)

Plot with \( K_c = \frac{p}{z} \)
LEAD MECHANICS

- MAXIMUM PHASE ADDED

\[ \sin \phi_{\text{max}} = \frac{1-\alpha}{1+\alpha} \quad \alpha = \frac{|z|}{|p|} \]

- FREQUENCY OF MAXIMUM PHASE ADDITION

\[ w_{\text{max}} = \sqrt{|z| \cdot |p|} \]

- HIGH FREQUENCY GAIN INCREASE \( \frac{1}{\alpha} \)

\[ \Rightarrow \text{COMPROMISE BETWEEN DESIRE FOR LARGE PHASE MARGIN (} \alpha \text{ SMALL) AND TENDENCY TO GENERATE LARGE GAINS AT HIGH FREQ.} \]

\[ \Rightarrow \text{KEEP } \frac{1}{\alpha} \leq 10 \quad (|p| \leq 10|z|) \]

\[ \Rightarrow \phi_{\text{max}} \leq 60^\circ \]

- USE MULTIPLE LEAD FILTERS TO GET MORE PHASE. \[ G_c = k_c \left( \frac{s+z_1}{s+p_1} \right) \left( \frac{s+z_2}{s+p_2} \right) \]

- SELECTION OF \( k_c \) IS PROBLEM SPECIFIC

  - COULD ARRANGE \[ |G_c(s)|_{s \to 0} = k_0 \]

  - OR SELECT \( k_c \) TO FORCE \( w_c = (w_c)_{\text{desired}} \)

- USE LEAD TO ADD PHASE \( \Rightarrow \) INCREASE PHASE MARGIN \( \Rightarrow \) IMPROVE TRANSIENT RESPONSE.
2. **LAG** \( |p| < |z| \)

- Pole at lower frequency than zero.
- \( \Rightarrow \) Gain decreases at high frequency.
- \( \Rightarrow \) Phase negative (i.e. adds lag).

\[
G_{\text{LAG}} = \frac{s/z + 1}{s/p + 1}
\]

- **LAG mechanics the same as for the lead.**
- Use **lag** to add 20 \( \log \alpha \) to low frequency gain with (hopefully) a small change to the phase margin.

\( \Rightarrow \) Plot shows gain much higher at low frequency.

\( \Rightarrow \) Keep lag dynamics typically well below the lead.
LEAD COMPENSATION: TYPICAL PROCESS

- Adding changes magnitude + phase
  ⇒ Difficult to predict new crossover frequency
  Hard to target $\phi_m$ at correct location.

Design approaches discussed in text.

$\phi_{req} = PM = (180 + \angle G(j\omega))$

- The process is slightly simpler if we target the lead compensator design only at the crossover frequency range.

  1) Find $\phi_{max}$ required ⇒ Find $\alpha = \frac{|z|}{|p|}$

  2) Put $\phi_{max}$ at crossover frequency
     $\omega_c^2 = |p| \cdot |z|$

  3) Select $k_c$ so crossover is at $\omega_c$
     ⇒ Meet bandwidth / phase margin specifications
     ⇒ No specified change in loop gain
        ⇒ Error constants.
EXAMPLE: \[ G(s) = \frac{1}{s(s+1)} \] would like \[ \omega_c = 10 \text{ rad/sec} \]
\[ \phi_m = 40^\circ \]

At 10 rad/s, slope \(-2\) \[ \Rightarrow \text{Plant phase} \approx 180^\circ \]

\[ \therefore \text{need to add a lead so that} \]

slope of the loop \[ G(s)G_c(s) \approx -1 \]

\[ \Rightarrow \text{adding a lead also increases the phase, giving us our } \phi_m \]

1. \[ \frac{Z}{P} = \frac{1 - \sin \phi_m}{1 + \sin \phi_m} \quad \Delta G(j\omega_c) = -180^\circ \]
\[ \phi_m = 40^\circ \]
\[ \phi_m = 40^\circ \]
\[ \therefore \frac{Z}{P} = 0.22 \]

2. \[ \omega_c^2 = Z \cdot P = 10^2 \quad \Rightarrow \quad Z = 4.7, \quad P = 21.4 \]

3. Pick \( K_c \) so \[ |G_cG| \bigg|_{s=10j} = 1 \]
% g=1/s/(s+1)
wc=10;num=1;den=conv([1 0],[1 1]);Phim=40*pi/180;
zdps=(1-sin(Phim))/(1+sin(Phim));z=sqrt(100*zdps);p=z/zdps;
numc=[1 z];denc=[1 p];
kc=abs(polyval(conv(den,denc),j*wc)/polyval(conv(num,numc),j*wc));
f=logspace(-2,2,400);g=freqresp(num,den,f,sqrt(-1));
ge=freqresp(kc,numc,denc,f,sqrt(-1));
loglog(f,abs(g),f,abs(gc),'-',f,abs(g.*ge),':');
title('Lead Example');xlabel('Freq (rad/sec)');ylabel('Magnitude')

dbode(num,den)
**LAG COMPENSATION**  **TYPICAL PROCESS**

- **Assumption is that we need to modify** (increase) the DC gain of the loop transfer function.

  ⇒ **If apply only a gain, then** $w_c$ **typically increases, and the phase margin decreases.**

  **This is not good** ⇒ **Use lag comp** to lower high freq. gain (or increase low freq. gain).

- **Analysis simpler with** $G_{lag} = \frac{k_c (s/z + 1)}{(s/p + 1)}$

1) **Pick** $|G_{lag}|_{s=0} = k_c$ **to give the desired low frequency gain for** $L = G_{lag}(s)G_p(s)$

2) **Pick** desired gain reduction at high frequency $-20\log\left(\frac{1}{\alpha}\right)$ ($\alpha = \frac{|z|}{|p|}$)

  ⇒ Normally pick $\alpha$ so that $w_c$ not changed.

3) **Heuristic:** want to limit frequency of the zero (+ pole) so that there is a minimal impact of the phase lag at $w = w_c$

  \[ \Rightarrow \text{Set } Z = \frac{w_c}{10} \]
**LAG DESIGN:**

**TWO WAYS TO GET DESIRED LOW FREQUENCY GAIN**

1. **USING JUST**
   
   \( K_c \) **INCREASES**
   
   \( \omega_c \) **AND LOWERS**
   
   OUR PM ↔ BAD

2. **ADD LOW FREQUENCY DYNAMICS (LAG)**
   
   THAT INCREASE LOW FREQUENCY GAIN, LEAVING GAIN NEAR + ABOVE ORIGINAL \( \omega_c \) **UNCHANGED**

- **MANY OTHER APPROACHES EXIST**
  
  → **TRY TO MEET ALL SPECS WITH ONLY ONE COMPENSATOR**

  → **APPROACH I OUTLINED Splits the Problem and Makes Each Step Easier**

  → **TYPICALLY REQUIRES BOTH LEAD AND LAG TO MEET ALL SPECS.**

- **SEE BOOK/HANDBOUT #9 FOR MORE EXAMPLES + RECIPES FOR OTHER DESIGN CASES.**