Topic #7

16.31 Feedback Control

State-Space Systems

- What are state-space models?
- Why should we use them?
- How are they related to the transfer functions used in classical control design and how do we develop a state-space model?
- What are the basic properties of a state-space model, and how do we analyze these?

Introduction

- State space model: a representation of the dynamics of an N^{th} order system as a first order differential equation in an N-vector, which is called the **state**.
  - Convert the N^{th} order differential equation that governs the dynamics into N first-order differential equations

- Classic example: second order mass-spring system

$$m\ddot{p} + c\dot{p} + kp = F$$

  - Let $x_1 = p$, then $x_2 = \dot{p} = \dot{x}_1$, and

$$\begin{aligned} \ddot{x}_2 = \ddot{p} &= (F - c\dot{p} - kp)/m \\ &= (F - cx_2 - kx_1)/m \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

  - Let $u = F$ and introduce the state

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \Rightarrow \dot{x} = Ax + Bu$$

  - If the measured output of the system is the position, then we have that

$$y = p = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = cx$$

- The most general continuous-time linear dynamical system has form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

where:

- $t \in \mathcal{R}$ denotes time
- $x(t) \in \mathcal{R}^n$ is the state (vector)
- $u(t) \in \mathcal{R}^m$ is the input or control
- $y(t) \in \mathcal{R}^p$ is the output

- $A(t) \in \mathcal{R}^{n \times n}$ is the dynamics matrix
- $B(t) \in \mathcal{R}^{n \times m}$ is the input matrix
- $C(t) \in \mathcal{R}^{p \times n}$ is the output or sensor matrix
- $D(t) \in \mathcal{R}^{p \times m}$ is the feedthrough matrix

- Note that the plant dynamics can be time-varying.
- Also note that this is a MIMO system.

- We will typically deal with the time-invariant case
$\Rightarrow$ **Linear Time-Invariant (LTI)** state dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

so that now A, B, C, D are constant and do not depend on t.

Basic Definitions

- **Linearity** – What is a linear dynamical system? A system G is linear with respect to its inputs and output

$$u(t) \rightarrow \boxed{G(s)} \rightarrow y(t)$$

if superposition holds:

$$G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 G u_1 + \alpha_2 G u_2$$

So if y_1 is the response of G to u_1 ($y_1 = Gu_1$), and y_2 is the response of G to u_2 ($y_2 = Gu_2$), then the response to $\alpha_1 u_1 + \alpha_2 u_2$ is $\alpha_1 y_1 + \alpha_2 y_2$

- A system is said to be **time-invariant** if the relationship between the input and output is independent of time. So if the response to $u(t)$ is $y(t)$, then the response to $u(t - t_0)$ is $y(t - t_0)$

- $x(t)$ is called the **state of the system** at t because:
  - Future output depends only on current state and future input
  - Future output depends on past input only through current state
  - State summarizes effect of past inputs on future output – like the *memory of the system*

- Example: Rechargeable flashlight – the state is the *current state of charge* of the battery. If you know that state, then you do not need to know how that level of charge was achieved (assuming a perfect battery) to predict the future performance of the flashlight.

Creating Linear State-Space Models

- Most easily created from N^{th} order differential equations that describe the dynamics
  - This was the case done before.
  - Only issue is which set of states to use – there are many choices.

- Can be developed from transfer function model as well.
  - Much more on this later

- Problem is that we have restricted ourselves here to linear state space models, and almost all systems are nonlinear in real-life.
  - Can develop linear models from nonlinear system dynamics

Linearization

- Often have a nonlinear set of dynamics given by

$$\dot{x} = f(x, u)$$

where x is once gain the state vector, u is the vector of inputs, and $f(\cdot,\cdot)$ is a nonlinear vector function that describes the dynamics

- **Example:** simple spring. With a mass at the end of a linear spring (rate k) we have the dynamics

$$m\ddot{x} = kx$$

but with a "leaf spring" as is used on car suspensions, we have a nonlinear spring – the more it deflects, the stiffer it gets. Good model now is

$$m\ddot{x} = (k_1 x + k_2 x^3)$$

which is a "cubic spring".

  – Restoring force depends on the deflection x in a nonlinear way.

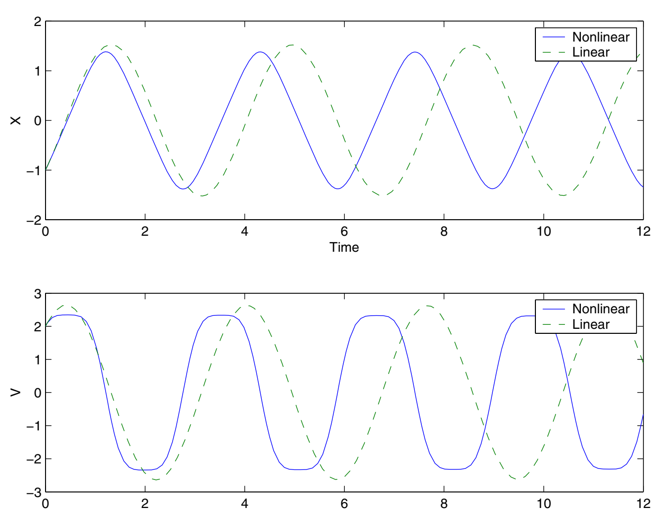

Figure 1: Response to linear k and nonlinear ($k_1 = 0, k_2 = k$) springs (code at the end)

- Typically assume that the system is operating about some nominal state solution $x^0(t)$ (possibly requires a nominal input $u^0(t)$)
  - Then write the actual state as $x(t) = x^0(t) + \delta x(t)$ and the actual inputs as $u(t) = u^0(t) + \delta u(t)$
  - The "δ" is included to denote the fact that we expect the variations about the nominal to be "small"

- Can then develop the linearized equations by using the **Taylor series expansion** of $f(\cdot, \cdot)$ about $x^0(t)$ and $u^0(t)$.

- Recall the vector equation $\dot{x} = f(x, u)$, each equation of which

$$\dot{x}_i = f_i(x, u)$$

  can be expanded as

$$\begin{aligned} \frac{d}{dt}(x_i^0 + \delta x_i) &= f_i(x^0 + \delta x, u^0 + \delta u) \\ &\approx f_i(x^0, u^0) + \left.\frac{\partial f_i}{\partial x}\right|_0 \delta x + \left.\frac{\partial f_i}{\partial u}\right|_0 \delta u \end{aligned}$$

  where

$$\frac{\partial f_i}{\partial x} = \left[\begin{array}{ccc} \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_n} \end{array} \right]$$

  and $\cdot|_0$ means that we should evaluate the function at the nominal values of x^0 and u^0.

- The meaning of "small" deviations now clear – the variations in δx and δu must be small enough that we can ignore the higher order terms in the Taylor expansion of $f(x, u)$.

- Since $\frac{d}{dt}x_i^0 = f_i(x^0, u^0)$, we thus have that

$$\frac{d}{dt}(\delta x_i) \approx \left.\frac{\partial f_i}{\partial x}\right|_0 \delta x + \left.\frac{\partial f_i}{\partial u}\right|_0 \delta u$$

- Combining for all n state equations, gives (note that we also set "≈" → "=") that

$$\begin{aligned}
\frac{d}{dt}\delta x &= \begin{bmatrix} \left.\frac{\partial f_1}{\partial x}\right|_0 \\ \left.\frac{\partial f_2}{\partial x}\right|_0 \\ \vdots \\ \left.\frac{\partial f_n}{\partial x}\right|_0 \end{bmatrix} \delta x + \begin{bmatrix} \left.\frac{\partial f_1}{\partial u}\right|_0 \\ \left.\frac{\partial f_2}{\partial u}\right|_0 \\ \vdots \\ \left.\frac{\partial f_n}{\partial u}\right|_0 \end{bmatrix} \delta u \\
&= A(t)\delta x + B(t)\delta u
\end{aligned}$$

where

$$A(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_0$$

and

$$B(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ & & \vdots & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_0$$

- Similarly, if the nonlinear measurement equation is $y = g(x, u)$, can show that, if $y(t) = y^0 + \delta y$, then

$$\begin{aligned}\delta y &= \begin{bmatrix} \left.\frac{\partial g_1}{\partial x}\right|_0 \\ \left.\frac{\partial g_2}{\partial x}\right|_0 \\ \vdots \\ \left.\frac{\partial g_p}{\partial x}\right|_0 \end{bmatrix} \delta x + \begin{bmatrix} \left.\frac{\partial g_1}{\partial u}\right|_0 \\ \left.\frac{\partial g_2}{\partial u}\right|_0 \\ \vdots \\ \left.\frac{\partial g_p}{\partial u}\right|_0 \end{bmatrix} \delta u \\ &= C(t)\delta x + D(t)\delta u\end{aligned}$$

- Typically think of these nominal conditions x^0, u^0 as "set points" or "operating points" for the nonlinear system. The equations

$$\begin{aligned}\frac{d}{dt}\delta x &= A(t)\delta x + B(t)\delta u \\ \delta y &= C(t)\delta x + D(t)\delta u\end{aligned}$$

  then give us a linearized model of the system dynamic behavior about these operating/set points.

- Note that if x^0, u^0 are constants, then the partial fractions in the expressions for A–D are all constant $\rightarrow$ **LTI linearized model.**

- One particularly important set of operating points are the **equilibrium points** of the system. Defined as the states & control input combinations for which

$$\dot{x} = f(x^0, u^0) \equiv 0$$

  provides n algebraic equations to find the equilibrium points.

Example

- Consider the nonlinear spring with (set $m = 1$)

$$\ddot{y} = k_1 y + k_2 y^3$$

gives us the nonlinear model ($x_1 = y$ and $x_2 = \dot{y}$)

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ k_1 y + k_2 y^3 \end{bmatrix} \Rightarrow \dot{x} = f(x)$$

- Find the equilibrium points and then make a state space model

- For the equilibrium points, we must solve

$$f(x) = \begin{bmatrix} \dot{y} \\ k_1 y + k_2 y^3 \end{bmatrix} = 0$$

which gives

$$\dot{y}^0 = 0 \quad \text{and} \quad k_1 y^0 + k_2 (y^0)^3 = 0$$

  – Second condition corresponds to $y^0 = 0$ or $y^0 = \pm\sqrt{-k_1/k_2}$, which is only real if k_1 and k_2 are opposite signs.

- For the state space model,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_0 = \begin{bmatrix} 0 & 1 \\ k_1 + 3k_2 (y)^2 & 0 \end{bmatrix}_0 = \begin{bmatrix} 0 & 1 \\ k_1 + 3k_2 (y^0)^2 & 0 \end{bmatrix}$$

and the linearized model is $\dot{\delta x} = A \delta x$

- For the equilibrium point $y = 0$, $\dot{y} = 0$

$$A_0 = \begin{bmatrix} 0 & 1 \\ k_1 & 0 \end{bmatrix}$$

which are the standard dynamics of a system with **just** a linear spring of stiffness k_1

  – Stable motion about $y = 0$ if $k_1 < 0$

- Assume that $k_1 = 1$, $k_2 = -1/2$, then we should get an equilibrium point at $\dot{y} = 0$, $y = \pm\sqrt{2}$, and since $k_1 + k_2(y^0)^2 = 0$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2k_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

are the dynamics of a stable oscillator about the equilibrium point

  – Will explore this in detail later

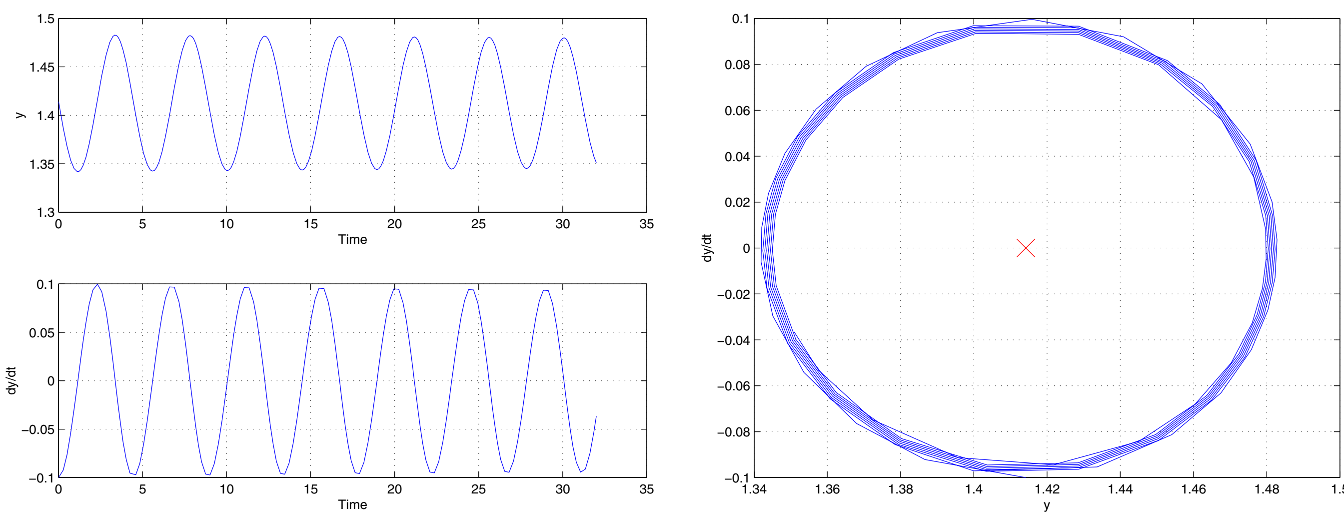

Figure 2: Nonlinear response ($k_1 = 1, k_2 = -.5$). The figure on the right shows the oscillation about the equilibrium point.

Linearized Nonlinear Dynamics

- Usually in practice we drop the "δ" as they are rather cumbersome, and (abusing notation) we write the state equations as:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{aligned}
$$

which is of the same form as the previous linear models

Example: Aircraft Dynamics

- Assumptions:
  1. Earth is an inertial reference frame
  2. A/C is a rigid body
  3. Body frame **B** fixed to the aircraft $(\vec{i}, \vec{j}, \vec{k})$

- The basic dynamics are:

$$\vec{F} = m\dot{\vec{v}}_c^{\,I} \quad \text{and} \quad \vec{T} = \dot{\vec{H}}^{I}$$

$$\Rightarrow \frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c \qquad \textbf{Transport Thm.}$$

$$\Rightarrow \vec{T} = \dot{\vec{H}}^{B} + {}^{BI}\vec{\omega} \times \vec{H}$$

- Instantaneous mapping of $\vec{v}_c$ and ${}^{BI}\vec{\omega}$ into the body frame is given by

$${}^{BI}\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k}$$

$$\Rightarrow {}^{BI}\omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

- The overall equations of motion are then:

$$\frac{1}{m}\vec{F} \;=\; \dot{\vec{v}}_c^{\,B} + \;^{BI}\vec{\omega} \times \vec{v}_c$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix}\begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

- These are clearly nonlinear – need to linearize about the equilibrium states.

- To find suitable equilibrium conditions, must solve

$$\frac{1}{m}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} - \begin{bmatrix} +QW - RV \\ +RU - PW \\ +PV - QU \end{bmatrix} = 0$$

- Assume steady state flight conditions with $\dot{P} = \dot{Q} = \dot{R} = 0$

- Define the **trim** angular rates, velocities, and Forces

$$ {}^{BI}\omega_B^o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad (v_c)_B^o = \begin{bmatrix} U_o \\ 0 \\ 0 \end{bmatrix} \qquad F_B^o = \begin{bmatrix} F_x^o \\ F_y^o \\ F_z^o \end{bmatrix} $$

  that are associated with the flight condition (they define the type of equilibrium motion that we analyze about).

  Note:

  – $W_0 = 0$ since we are using the stability axes, and

  – $V_0 = 0$ because we are assuming symmetric flight

- Can now linearize the equations about this flight mode. To proceed, define

$$
\begin{array}{lrccc}
\text{Velocities} & U_0, & U = U_0 + u & \Rightarrow & \dot{U} = \dot{u} \\
 & W_0 = 0, & W = w & \Rightarrow & \dot{W} = \dot{w} \\
 & V_0 = 0, & V = v & \Rightarrow & \dot{V} = \dot{v} \\
 & & & & \\
\text{Angular Rates} & P_0 = 0, & P = p & \Rightarrow & \dot{P} = \dot{p} \\
 & Q_0 = 0, & Q = q & \Rightarrow & \dot{Q} = \dot{q} \\
 & R_0 = 0, & R = r & \Rightarrow & \dot{R} = \dot{r} \\
 & & & & \\
\text{Angles} & \Theta_0, & \Theta = \Theta_0 + \theta & \Rightarrow & \dot{\Theta} = \dot{\theta} \\
 & \Phi_0 = 0, & \Phi = \phi & \Rightarrow & \dot{\Phi} = \dot{\phi} \\
 & \Psi_0 = 0, & \Psi = \psi & \Rightarrow & \dot{\Psi} = \dot{\psi}
\end{array}
$$

- **Linearization for symmetric flight**
  $U = U_0 + u$, $V_0 = W_0 = 0$, $P_0 = Q_0 = R_0 = 0$.
  Note that the forces and moments are also perturbed.

$$\begin{aligned}
\frac{1}{m}\left[F_x^0 + \Delta F_x\right] &= \dot{U} + QW - RV \approx \dot{u} + qw - rv \\
&\approx \dot{u}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{m}\left[F_y^0 + \Delta F_y\right] &= \dot{V} + RU - PW \approx \dot{v} + r(U_0 + u) - pw \\
&\approx \dot{v} + rU_0
\end{aligned}$$

$$\begin{aligned}
\frac{1}{m}\left[F_z^0 + \Delta F_z\right] &= \dot{W} + PV - QU \approx \dot{w} + pv - q(U_0 + u) \\
&\approx \dot{w} - qU_0
\end{aligned}$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} \Delta F_x \\ \Delta F_y \\ \Delta F_z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} + rU_0 \\ \dot{w} - qU_0 \end{bmatrix}$$

- Which gives the linearized dynamics for the aircraft motion about the steady-state flight condition.
  - Need to analyze the perturbations to the forces and moments to fully understand the linearized dynamics – take 16.61
  - Can do same thing for the rotational dynamics.

```
% save this entire code as plant.m
%
function [xdot] = plant(t,x)
global n
xdot(1) = x(2);
xdot(2) = -3*(x(1))^n;
xdot = xdot';
return

% the use this part of the code in Matlab®
% to call_plant.m
global n
n=3; %nonlinear
x0 = [-1 2]; % initial condition
[T,x]=ode23('plant', [0 12], x0); %simulate NL equations for 12 sec
n=1; % linear
[T1,x1]=ode23('plant', [0 12], x0);

subplot(211)
plot(T,x(:,1),T1,x1(:,1),'--');
legend('Nonlinear','Linear')
ylabel('X')
xlabel('Time')
subplot(212)
plot(T,x(:,2),T1,x1(:,2),'--');
legend('Nonlinear','Linear')
ylabel('V')
xlabel('Time')
```