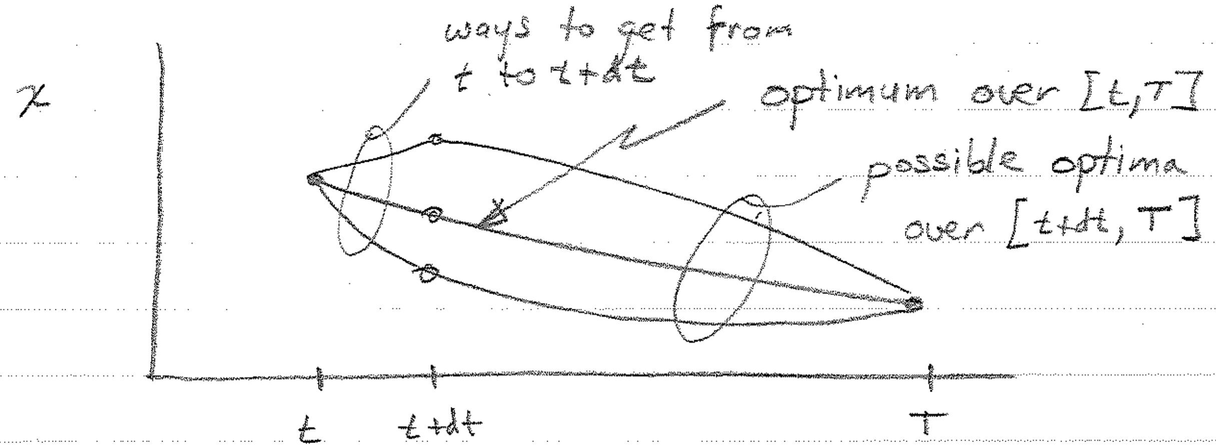
Lecture 21

Solution by Dynamic Programming

Principle of optimality:

> Any portion of an optimal trajectory is an optimal trajectory.



Optimal "cost-to-go" is

$$J^*(x(t),t) = \min_{u(\cdot)} \int_t^T \left[x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\right]d\tau$$

$$= \min_{u(t)} \Bigg(\underbrace{\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt}_{\text{cost over } [t,t+dt]} + \underbrace{J^*(x(t+dt), t+dt)}_{\text{cost over } [t+dt,T]} \Bigg)$$

where

$$x(t+dt) = x(t) + \left[A(t)x(t) + B(t)u(t)\right]dt$$

If we know $J^*(x, t+dt)$ for all x, can choose best u at time t.

Let's <u>guess</u> that $J^*(x,t) = x^T P(t) x$

need to find!

(Because J is quadratic)

Then

$$x^T(t) P(t) x(t) =$$

$$\min_u \Big\{ [x^T(t) Q(t) x(t) + u^T(t) R(t) u(t)] dt + (x(t) + [A(t)x(t) + B(t)u(t)]dt)^T \cdot \overbrace{P(t+dt)}^{P + \dot{P} dt} \cdot (x(t) + [A(t)x(t) + B(t)u(t)]dt] \Big\}$$

Keep only $O(1)$ and $O(dt)$ terms:

$$x^T P x =$$

$$\min_u \Big\{ [x^T Q x + u^T R u] dt + x^T P x + x^T \dot{P} x \, dt + x^T P (Ax + Bu) dt + (Ax + Bu)^T P x \, dt \Big\}$$

To minimize,

$$\frac{d}{du} \{ \cdot \} = 0 = (2Ru + 2B^T P x) dt$$

$$\Rightarrow \quad u = -R^{-1}B^T P x$$

is the optimum control (if form of J is correct)

$$u(t) = -R^{-1}(t)\,B^T(t)\,P(t)\,x(t) = -F(t)\,x(t)$$

$$\boxed{F(t) = R^{-1}(t)\,B^T(t)\,P(t)}$$

So,

$$\cancel{x^T P x} = \big(x^T Q x + x^T P B R^{-1} R R^{-1} B^T P x\big)\,dt + \cancel{x^T P x}$$
$$+\, x^T \dot{P} x\,dt + x^T P\,(Ax - BR^{-1}B^T P x)\,dt$$
$$+\, (Ax - BR^{-1}B^T P x)^T P x\,dt\;\big)$$

$$0 = x^T\big(Q + PBR^{-1}B^TP + \dot{P} + PA - PBR^{-1}B^TP$$
$$+\, A^TP - PBR^{-1}B^TP\big)\,x$$

Therefore, $P(t)$ satisfies

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q(t)$$
$$- P(t)B(t)R^{-1}B(t)P(t)$$

"Riccati Equation"

Integrate <u>backwards</u> in time. Final condition: $P(T) = 0$

The Steady-State Solution

In many cases

- $T = \infty$ (or T large)
- A, B, Q, R constant

In this case, expect $P(t) \to$ const. as $T - t \to \infty$.

If $P(t)$ reaches a steady state, P satisfies the "algebraic Riccati equation" (ARE)

$$0 = A^T P + PA + Q - PBR^{-1}B^T P$$

(Like a quadratic eq'n) $\Rightarrow$ more than 1 solution

and the optimal gain is

$$F = R^{-1}B^T P$$

Theorem If (A, B) is stabilizable and $(A, Q^{1/2})$ is detectable,

$$\lim_{T-t \to \infty} P(t) = \bar{P} \quad \text{exists,}$$

which is the unique solution of the ARE for which $P \geq 0$.

Lecture 22

Note: Didn't go well. See Lecture 23

The Steady-State Solution

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which is the unique solution of the ARE for which $P > 0$.

Solving the Riccati Equation

Riccati equation is intimately related to Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}}_{\text{Hamiltonian matrix, } H.} \begin{pmatrix} x \\ p \end{pmatrix}$$

why?

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1) Could solve optimal control problem by calculus of variations. p is the Lagrange multiplier, and can show that $p = Px$

2) Define $p = Px$, and see what happens.

$$\dot{x} = Ax + Bu = Ax - BFx$$

$$= Ax - BR^{-1}B^T \underbrace{Px}_{p}$$

$$\boxed{\dot{x} = Ax - BR^{-1}B^T p}$$

$$\dot{p} = \frac{d}{dt}(Px) = \dot{P}x + P\dot{x}$$

$$= -(A^TP + \cancel{PA} + Q - \cancel{PBR^{-1}B^TP})x + P(\cancel{Ax} + \cancel{Bu})$$

$$\boxed{\dot{p} = -Qx - A^TPx = -Qx - A^Tp}$$

So $\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$

~~Riccati~~ equation $\Rightarrow$ Hamiltonian.

How do we go the other way?

Can show that $\phi_H(s) = \det(sI - H) = \phi_H(-s)$
$\Rightarrow$ if λ is an eigenvalue of H, so is $-\lambda$.

<u>Proof</u> Take $T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Then

$$THT^{-1} = \begin{bmatrix} -A^T & Q \\ +BR^{-1}B^T & A \end{bmatrix} = -H^T$$

$$\Rightarrow \lambda(H) = \lambda(THT^{-1}) = \lambda(-H^T) = -\lambda(H) = -\lambda(H)$$

So <u>half</u> the poles of H are stable. These must correspond to stable regulator poles.

So,

1) Find the eigenvalues, eigenvectors of H.
2) Keep only the stable ones.
3) Let $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$