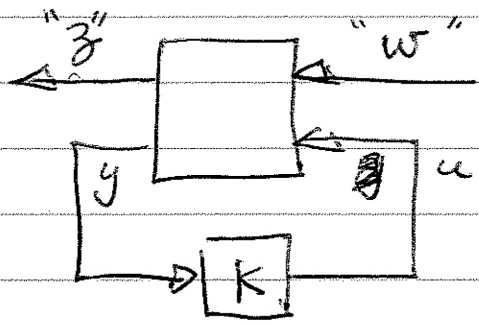
H_∞ control

Problem: Find a controller, $K(s)$, so that



$$\|T_{zw}\|_\infty < \gamma$$

Note: "z" includes z and u; "w" includes w and v. So cost function is like

$$J = \int \left(|z|^2 + |u|^2 - \gamma^2 \left[|w|^2 + |v|^2 \right] \right) dt$$

To solve, do

(1) Full information control

(2) Estimation

(3) Combined controller.

$$J = \min_u \max_w \int \left(|z|^2 + |u|^2 - \gamma^2 |w|^2 \right) dt$$

Assume a cost-to-go of form

$$J = x^T P x$$

Then applying dynamic programming yields

$$u = -B_u^T P x + \frac{B_w^T P x}{\gamma^2}$$

The Riccati equation for P is given by

$$-\dot{P} = A^T P + PA + C_z^T C_z + \gamma^{-2} P B_w B_w^T P - P B_u B_u^T P$$

$$= 0 \quad \text{(in steady-state)}$$

ARE has a solution iff

$$H = \begin{pmatrix} A & -B_u B_u^T + \gamma^{-2} B_w B_w^T \\ -C_z^T C_z & -A^T \end{pmatrix}$$

has no poles on $j\omega$-axis.

Note that the F.S. controller is

$$u = -Fx$$

$$F = B_u^T P$$

H∞ estimator.

The dual estimation problem yields

$$\dot{\Sigma} = A\Sigma + \Sigma A^T + B_w B_w^T - \Sigma C_y^T C_y \Sigma + \gamma^{-2} \Sigma C_z^T C_z \Sigma$$
$$= 0 \text{ in steady-state.}$$

The corresponding Hamiltonian is

$$J = \begin{bmatrix} A^T & \gamma^{-2} C_z^T C_z - C_y^T C_y \\ -B_w B_w^T & -A \end{bmatrix}$$

The Riccati equation has a solution iff J has no $j\omega$-axis poles.

The H∞ estimator is

$$\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x})$$

$$L = -\Sigma C_y^T$$

What is interpretation of Σ?
The estimation error is bounded by

$$\int_{-\infty}^{t} \left(|z|^2 - \gamma^2 \left[|w|^2 + |v|^2 \right] \right) dt$$

$$< - (x - \hat{x})^T \Sigma^{-1} (x - \hat{x}) \gamma^2$$

Can w, v drive system unstable?
Suppose for $t < 0$, w, v are such that $y \equiv 0$. Then $\hat{x} \equiv 0$, so there is no control ($u \equiv 0$). Then

$$\int_{-\infty}^{0} \left(|z|^2 + |u|^2 - \gamma^2 \left(|w|^2 + |v|^2 \right) \right) dt$$

$$< - x^T \Sigma^{-1} x \, \gamma^2$$

After $t = 0$, assume we know state, so do perfect control. Then

$$\int_0^\infty \left(z^2 + u^2 - \gamma^2 w^2 - \gamma^2 v^2 \right) dt <$$

$$\int_0^\infty \left(z^2 + u^2 - \gamma^2 w^2 \right) dt < x^T P x$$

So total cost J satisfies

$$J < x^T P x - x^T \Sigma^{-1} x \gamma^2$$

if $P < \Sigma^{-1}\gamma^2$ $J < 0 \Rightarrow \|T_{zw}\| < \gamma$ (sort of). But if $P > \Sigma^{-1}$, can find $x(0)$ for which cost is $+\infty$!

So requirement for H_∞ stabilizing controller is

$$P < \Sigma^{-1}\gamma^2$$

which is the same as

$$\rho(\Sigma P) < \gamma^2$$

"spectral radius"

So estimation and control problems are <u>coupled</u>! There is not a <u>nice</u> separation principle.

H_∞ controller is given by

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\hat{A}_\infty = A + \gamma^{-2} B_\omega B_\omega^T P - B_2 F_\infty + Z_\infty L_\infty C_y$$

$$F_\infty = B_u^T P$$

$$L_\infty = \Sigma C_y^T$$

$$Z_\infty = (I - \gamma^{-2} \Sigma P)^{-1}$$

Can find best controller by γ-iteration.