Kalman Filter

[Discuss Wiener filter, its important role, but lack of generality — works only for stationary systems]

[Kalman-Bucy developed Kalman filter in ~1960 ← revolutionized the way things are done.]

[Kalman filter was inevitable — Prof. Battin had (some of) the equations. The state-space method begs us to see the answer]

Consider the system

$$\dot{x} = Ax + Gw \qquad w \sim W, \text{ white}$$

$$y = Cx + v \qquad v \sim V, \text{ white independent}$$

Assume an observer of the form

$$\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$$

What is the "best" observer?

The observer error is

$$e = x - \hat{x}$$

$$\Rightarrow \dot{e} = Ax + Gw - A\hat{x} - K(Cx + v - C\hat{x})$$
$$= (A - KC)e + \underbrace{Gw - Kv}_{\text{noise driving } e(t)}$$

How to choose K?

K large $\Rightarrow$ A-KC "fast," so effect of noise is attenuated

K small $\Rightarrow$ Kv noise is small — less to attenuate.

What is best choice? Must choose a metric, e.g.,

$$J = E[e^T S e]$$

positive definite

$$= E[tr[See^T]] = tr[SP]$$

covariance of e

The covariance satisfies

$$\dot{P} = (A-KC)P + P(A-KC)^T + GWG^T + KVK^T$$

$$J = tr[SP] = \int tr[S\dot{P}]\,dt$$

So to make J small, make $\dot{J}$ small

[This isn't really quite right, but its close to true! It works because of white noise assumption]

Minimize $\dot{J} = tr(S\dot{P})$:

$$\dot{J} = tr\left[S\left((A-KC)P + P(A-KC)^T + GWG^T + KVK^T\right)\right]$$

Use $\frac{d}{dB} tr(ABC) = A^T C^T$

$$\frac{d\dot{J}}{dK} = -2SPC^T + 2SKV = 0$$

$\Rightarrow K = PC^TV^{-1}$ Independent of S!

For this K,

$$\dot{P} = (A - PC^T V^{-1} C) P + P (A^T - C^T V^{-1} C P) + GWG^T + PC^T V^{-1} V V^{-1} C P$$

$$\boxed{\dot{P} = AP + PA^T + GWG^T - PC^T V^{-1} C P}$$

"Riccati equation"

Note the strong parallel to the LQR problem.

Kalman Filter is the <u>dual</u> of the LQR

Note:
- Solution is more difficult when w and v are correlated.

<u>steady-state Kalman Filter</u>

When the problem is LTI and stationary, the steady-state covariance should satisfy

$$0 = AP + PA^T + GWG^T - PC^T V^{-1} C P$$

"Algebraic Riccati Equation"