The Linear Quadratic Gaussian (LQG) Problem

We are now ready to solve the real problem:

Find the compensator for the system

$$\dot{x} = Ax + Bu + Gw$$
$$y = Cx + v$$
$$z = Ex$$

where $w(t)$, $v(t)$ are white, that minimizes the cost

$$J = E\left[\int_{t_1}^{t_2} (z^T Q z + u^T R u)\,dt\right]$$

or for the stationary problem,

$$J = E\left[z^T Q z + u^T R u\right]$$

The obvious solution

LQR + Kalman Filter

is correct! (But must prove this)

Conclusion: Separation principle always allows a controllable, observable system to have closed-loop poles @ any desired location!

Note that there is no guarantee that

1. Compensator will be stable

2. System will be "robust"

Example $\quad \dot{x} = -x + u \qquad G(s) = \dfrac{1}{s+1}$

$y = x$

Find compensator to place poles at $s = -3$, $s = -3$:

$$a - kc = -3$$
$$= -1 - 1 \cdot k \Rightarrow k = 2$$

$$a - bf = -3$$
$$= -1 - 1 \cdot f \Rightarrow f = +2$$

Compensator is

$$K(s) = (a - bf - kc,\ k,\ -f)$$

$$= (-5,\ 2,\ -2)$$

$$= \frac{-4}{s+5}$$

Check:

$$1 - K(s)G(s) = 1 + \frac{4}{(s+1)(s+5)}$$

$$\Rightarrow \Delta(s) = (s+1)(s+5) + 4$$
$$= s^2 + 6s + 9 = (s+3)^2 \checkmark$$

LQG Solution (Informal)

What is additional cost due to making an error in control?

$$J = (x^T Q x + u^T R u)\, dt + J^*(X(t+dt), t+dt)$$

$$= (x^T Q x + u^T R u)\, dt + x^T P x + x^T \dot{P} x\, dt + x^T P (Ax + Bu)\, dt + (Ax + Bu)^T P x\, dt$$

Let $u = u^* + \delta u = -R^{-1} B^T P x + \delta u$. Then

$$J = \delta u^T R\, \delta u\, dt + x^T P x = \delta u^T R\, \delta u\, dt + J^*$$

Therefore, the additional cost is

$$\text{extra cost} = \int E\left(\delta u^T R\, \delta u\right) dt$$

$$= \int tr\left[R F \Sigma F^T \right] dt$$

since $\delta u = -\cancel{F}^{K}(\hat{x} - x) = \cancel{Fe}\, Ke$

This is the additional cost due to estimation error. The additional cost due to process noise is

$$\int E\left[w^T G^T P G w\right] dt$$

$$= \int tr\left[G^T P G W\right] dt$$

Therefore, the LQG cost is

$$J = \int_0^T tr\left[\overset{RK\Sigma K^T}{\cancel{RK\Sigma K}^T} + G^T P G W\right] dt$$

Can be shown that also

$$J = \int_0^T tr\left[\Sigma E^T Q E + P L V L^T\right] dt$$

This follows from <u>duality</u>.