Topic #10

16.31 Feedback Control

State-Space Systems

- **What are the basic properties of a state-space model, and how do we analyze these?**
- Time Domain Interpretations
- System Modes

- **Forced Solution**
  - Consider a **scalar case:**

$$\begin{aligned}\dot{x} &= ax + bu, \quad x(0) \text{ given} \\ \Rightarrow x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau\end{aligned}$$

  where did this come from?

  1. $\dot{x} - ax = bu$
  2. $e^{-at}\left[\dot{x} - ax\right] = \frac{d}{dt}(e^{-at}x(t)) = e^{-at}bu(t)$
  3. $\int_0^t \frac{d}{d\tau} e^{-a\tau} x(\tau) d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau} bu(\tau) d\tau$

- **Forced Solution – Matrix case:**

$$\dot{x} = Ax + Bu$$

where x is an n-vector and u is a m-vector

- Just follow the same steps as above to get

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

and if $y = Cx + Du$, then

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

  - $Ce^{At}x(0)$ is the initial response
  - $Ce^{A(t)}B$ is the impulse response of the system.

- Have seen the key role of e^{At} in the solution for $x(t)$
  - Determines the system time response
  - But would like to get more insight!
- Consider what happens if the matrix A is diagonalizable, i.e. there exists a T such that

$$T^{-1}AT = \Lambda \text{ which is diagonal } \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Then

$$e^{At} = Te^{\Lambda t}T^{-1}$$

where

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

- Follows since $e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$ and that $A = T\Lambda T^{-1}$, so we can show that

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \ldots \\ &= I + T\Lambda T^{-1}t + \frac{1}{2!}(T\Lambda T^{-1}t)^2 + \ldots \\ &= Te^{\Lambda t}T^{-1} \end{aligned}$$

- This is a simpler way to get the matrix exponential, but how find T and λ?
  - Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

- Recall that the eigenvalues of A are the same as the roots of the characteristic equation (page 9–1)
- λ is an **eigenvalue** of A if
$$\det(\lambda I - A) = 0$$
which is true iff there exists a nonzero v (**eigenvector**) for which
$$(\lambda I - A)v = 0 \quad \Rightarrow \quad Av = \lambda v$$
- Repeat the process to find all of the eigenvectors. Assuming that the n eigenvectors are linearly independent
$$\begin{aligned} Av_i &= \lambda_i v_i \quad i = 1, \ldots, n \\ A \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] &= \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right] \\ AT = T\Lambda \quad \Rightarrow & \qquad T^{-1}AT = \Lambda \end{aligned}$$
- One word of caution: Not all matrices are diagonalizable
$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \qquad \det(sI - A) = s^2$$
only one eigenvalue $s = 0$ (repeated twice). The eigenvectors solve
$$\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} r_1 \\ r_2 \end{array} \right] = 0$$
eigenvectors are of the form $\left[\begin{array}{c} r_1 \\ 0 \end{array} \right]$, $r_1 \neq 0 \rightarrow$ would only be one.
- Need the **Jordan Normal Form** to handle this case (section 3.7.3)

Mechanics

- Consider $A = \begin{bmatrix} -1 & 1 \\ -8 & 5 \end{bmatrix}$

$$(sI - A) = \begin{bmatrix} s+1 & -1 \\ 8 & s-5 \end{bmatrix}$$

$$\det(sI - A) = (s+1)(s-5) + 8 = s^2 - 4s + 3 = 0$$

so the eigenvalues are $s_1 = 1$ and $s_2 = 3$

- Eigenvectors $(sI - A)v = 0$

$$(s_1 I - A)v_1 = \left.\begin{bmatrix} s+1 & -1 \\ 8 & s-5 \end{bmatrix}\right|_{s=1} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \qquad 2v_{11} - v_{21} = 0, \Rightarrow v_{21} = 2v_{11}$$

v_{11} is then arbitrary ($\neq 0$), so set $v_{11} = 1$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(s_2 I - A)v_2 = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \qquad 4v_{12} - v_{22} = 0, \Rightarrow v_{22} = 4v_{12}$$

$$v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- Confirm that $Av_i = \lambda_i v_i$

Dynamic Interpretation

- Since $A = T\Lambda T^{-1}$, then

$$e^{At} = Te^{\Lambda t}T^{-1} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

where we have written

$$T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

which is a column of rows.

- Multiply this expression out and we get that

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T$$

- Assume A diagonalizable, then $\dot{x} = Ax$, $x(0)$ given, has solution

$$\begin{aligned} x(t) &= e^{At}x(0) = Te^{\Lambda t}T^{-1}x(0) \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \{w_i^T x(0)\} \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \beta_i \end{aligned}$$

- State solution is a **linear combination** of the system modes $v_i e^{\lambda_i}$

$e^{\lambda_i t}$ – Determines the **nature** of the time response

v_i – Determines extent to which each state **contributes** to that mode

β_i – Determines extent to which the initial condition **excites** the mode

- Note that the v_i give the relative sizing of the response of each part of the state vector to the response.

$$v_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \quad \text{mode 1}$$

$$v_2(t) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} e^{-3t} \quad \text{mode 2}$$

- Clearly $e^{\lambda_i t}$ gives the time modulation
  - λ_i real – growing/decaying exponential response
  - λ_i complex – growing/decaying exponential damped sinusoidal

- **Bottom line:** The locations of the eigenvalues determine the pole locations for the system, thus:
  - They determine the stability and/or performance & transient behavior of the system.
  - **It is their locations that we will want to modify when we start the control work**

Zeros in State Space Models

- Roots of the transfer function numerator are called the system zeros.
  - Need to develop a similar way of defining/computing them using a state space model.

- **Zero:** is a generalized frequency s_0 for which the system can have a non-zero input $u(t) = u_0 e^{s_0 t}$, but exactly zero output $y(t) \equiv 0 \ \forall t$
  - Note that there is a specific initial condition associated with this response x_0, so the state response is of the form $x(t) = x_0 e^{s_0 t}$

$$u(t) = u_0 e^{s_0 t} \quad \Rightarrow \quad x(t) = x_0 e^{s_0 t} \quad \Rightarrow y(t) \equiv 0$$

- Given $\dot{x} = Ax + Bu$, substitute the above to get:

$$x_0 s_0 e^{s_0 t} = A x_0 e^{s_0 t} + B u_0 e^{s_0 t} \quad \Rightarrow \quad \begin{bmatrix} s_0 I - A & -B \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

- Also have that $y = Cx + Du = 0$ which gives:

$$C x_0 e^{s_0 t} + D u_0 e^{s_0 t} = 0 \quad \rightarrow \quad \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

- So we must solve for the s_0 that solves: or

$$\begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

  - This is a **generalized eigenvalue problem** that can be solved in MATLAB® using `eig.m` or `tzero.m` [2]

[2] MATLAB® is a trademark of the Mathworks Inc.

- Is a zero at the frequency s_0 if there exists a non-trivial solution of

$$\det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = 0$$

  - Compare with equation on page 9–1

- **Key Point:** Zeros have both a direction $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$ and a frequency s_0
  - Just as we would associate a direction (eigenvector) with each pole (frequency λ_i)

- Example: $G(s) = \frac{s+2}{s^2+7s+12}$

$$A = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad D = 0$$

$$\det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = \det \begin{bmatrix} s_0+7 & 12 & -1 \\ -1 & s_0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= (s_0+7)(0) + 1(2) + 1(s_0) = s_0 + 2 = 0$$

so there is clearly a zero at $s_0 = -2$, as we expected. For the directions, solve:

$$\begin{bmatrix} s_0+7 & 12 & -1 \\ -1 & s_0 & 0 \\ 1 & 2 & 0 \end{bmatrix}_{s_0=-2} \begin{bmatrix} x_{01} \\ x_{02} \\ u_0 \end{bmatrix} = \begin{bmatrix} 5 & 12 & -1 \\ -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ u_0 \end{bmatrix} = 0?$$

gives $x_{01} = -2x_{02}$ and $u_0 = 2x_{02}$ so that with $x_{02} = 1$

$$x_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad u = 2e^{-2t}$$

- Further observations: apply the specified control input in the frequency domain, so that
$$Y_1(s) = G(s)U(s)$$
where $u = 2e^{-2t}$, so that $U(s) = \frac{2}{s+2}$
$$Y_1(s) = \frac{s+2}{s^2+7s+12} \cdot \frac{2}{s+2} = \frac{2}{s^2+7s+12}$$
Say that $s = -2$ is a **blocking zero** or a **transmission zero.**

- The response $Y(s)$ is clearly non-zero, but it does not contain a component at the input frequency $s = -2$. That input has been "blocked".

- Note that the output response left in $Y_1(s)$ is of a very special form – it corresponds to the (negative of the) response you would see from the system with $u(t) = 0$ and $x_0 = \begin{bmatrix} -2 & 1 \end{bmatrix}^T$
$$\begin{aligned} Y_2(s) &= C(sI - A)^{-1}x_0 \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s+7 & 12 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s & -12 \\ 1 & s+7 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{s^2+7s+12} \\ &= \frac{-2}{s^2+7s+12} \end{aligned}$$

- So then the total output is $Y(s) = Y_1(s) + Y_2(s)$ showing that $Y(s) = 0 \rightarrow y(t) = 0$, as expected.

- **Summary of Zeros:** Great feature of solving for zeros using the generalized eigenvalue matrix condition is that it can be used to find **MIMO zeros** of a system with multiple inputs/outputs.

$$\det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = 0$$

- Need to be very careful when we find MIMO zeros that have the same frequency as the poles of the system, because it is not obvious that a pole/zero cancellation will occur (for MIMO systems).
  – The zeros have a directionality associated with them, and that must "agree" as well, or else you do not get cancellation
  – More on this topic later.

- Relationship to **transfer function matrix:**
  – If z is a zero with (right) direction $[\zeta^T \ , \ \tilde{u}^T \ ]^T$, then

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \zeta \\ \tilde{u} \end{bmatrix} = 0$$

  – If z not an eigenvalue of A, then $\zeta = (zI - A)^{-1} B\tilde{u}$, which gives

$$\left[C(zI - A)^{-1}B + D\right] \tilde{u} = G(z)\tilde{u} = 0$$

  – Which implies that $G(s)$ loses rank at $s = z$
  – If $G(s)$ is square, can test: $\det \ \mathbf{G(s) = 0}$
  – If any of the resulting roots are also eigenvalues of A, need to re-check the generalized eigenvalue matrix condition.

- Note that the *transfer function matrix* (TFM) notion is a MIMO generalization of the SISO transfer function
  - It is a matrix of transfer functions

$$G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ & \ddots & \\ g_{p1}(s) & \cdots & g_{pm}(s) \end{bmatrix}$$

  - where $g_{ij}(s)$ relates the input of actuator j to the output of sensor i.

- Example:

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \\ \dfrac{1}{s-2} & \dfrac{s-2}{s+2} \end{bmatrix}$$

- It is relatively easy to go from a state-space model to a TFM, but not obvious how to go back the other way.

- Note: we have to be careful how to analyze these TFM's.
  - Just looking at the individual transfer functions is **not useful.**
  - Need to look at the system as a whole – will develop a new tool based on the **singular values** of $G(s)$