Topic #9

16.31 Feedback Control

State-Space Systems

- **What are the basic properties of a state-space model, and how do we analyze these?**
- SS to TF

SS ⇒ TF

- In going from the state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

  to the transfer function $G(s) = C(sI - A)^{-1}B + D$ need to form the inverse of the matrix $(sI - A)$ – a symbolic inverse – not easy at all.

- For simple cases, we can use the following:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}^{-1} = \frac{1}{a_1 a_4 - a_2 a_3} \begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}$$

  For larger problems, we can also use *Cramer's Rule*

- Turns out that an equivalent method is to form:

$$G(s) = C(sI - A)^{-1}B + D = \frac{\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}}{\det(sI - A)}$$

  – Reason for this will become more apparent later when we talk about how to compute the "zeros" of a state-space model (which are the roots of the numerator)

- Example from before:

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$$

  then

$$G(s) = \frac{1}{\det(sI - A)} \left[\begin{array}{ccc|c} s + a_1 & a_2 & a_3 & -1 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ \hline b_1 & b_2 & b_3 & 0 \end{array} \right] = \frac{b_3 + b_2 s + b_1 s^2}{\det(sI - A)}$$

  and $\det(sI - A) = s^3 + a_1 s^2 + a_2 s + s_3$

- **Key point:** Characteristic equation of this system given by $\det(sI - A)$

Time Response

- Can develop a lot of insight into the system response and how it is modeled by computing the time response $x(t)$
  - Homogeneous part
  - Forced solution
- **Homogeneous Part**
$$\dot{x} = Ax, \quad x(0) \text{ known}$$
  - Take Laplace transform
$$X(s) = (sI - A)^{-1}x(0)$$
    so that
$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]x(0)$$
  - But can show
$$\begin{aligned}(sI - A)^{-1} &= \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots \\ \text{so } \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] &= I + At + \frac{1}{2!}(At)^2 + \ldots \\ &= e^{At}\end{aligned}$$
  - So
$$x(t) = e^{At}x(0)$$
- e^{At} is a special matrix that we will use many times in this course
  - *Transition matrix*
  - *Matrix Exponential*
  - Calculate in MATLAB® using `expm.m` and not `exp.m` [1]
  - Note that $e^{(A+B)t} = e^{At}e^{Bt}$ iff $AB = BA$
- We will say more about e^{At} when we have said more about A (eigenvalues and eigenvectors)
- Computation of $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$ straightforward for a 2-state system

[1] MATLAB® is a trademark of the Mathworks Inc.

- Example: $\dot{x} = Ax$, with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{(s+2)(s+1)} \\ &= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix} \end{aligned}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$