Random Processes

A better model for a dynamic system:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + G(t)\,w(t)$$
$$y(t) = C(t)\,x(t) + v(t)$$

$w(t)$ = "process noise"
$v(t)$ = "measurement noise"

For example,

Plant = aircraft
$w(t)$ = turbulence
$v(t)$ = gyro noise

$w(t)$, $v(t)$ are <u>random processes</u>

A random process $w(t)$ is described by its joint probability density functions

$$p(\omega; t)$$

$$p(\omega_1, \omega_2; t_1, t_2)$$

$$p(\omega_1, \omega_2, \omega_3; t_1, t_2, t_3)$$

$$\vdots$$

E.g.,

$$p(\omega_1, \omega_2; t_1, t_2)\, \Delta\omega_1\, \Delta\omega_2$$

$$= \text{Prob}\left[\omega_1 < \omega(t_1) < \omega_1 + \Delta\omega_1,\ \omega_2 < \omega(t_2) < \omega_2 + \Delta\omega_2\right]$$

In most cases, we care only about first and second order statistics:

mean:

$$\bar{\omega}(t) = E[\omega(t)] = \int_{-\infty}^{\infty} \omega\, p[\omega; t]\, d\omega$$

mean square:

$$E[\omega^2(t)] = \int_{-\infty}^{\infty} \omega^2 p^2[\omega; t]\, d\omega$$

Variance:

$$\sigma^2(t) = E\left[(\omega(t) - \bar{\omega}(t))^2\right]$$

Correlation function:

$$\rho(t) = E[\omega(t)\,\omega(\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega_1 \omega_2\, p[\omega_1, \omega_2; t_1, t_2] \cdot d\omega_1\, d\omega_2$$

For vector processes,

$$R(t, \tau) = E\left[\omega(t)\, \omega^T(\tau)\right]$$

$$= \text{"correlation matrix"}$$

$$R(t, t) = E\left[\omega(t)\, \omega^T(t)\right] = \text{"covariance matrix"}$$

Stationary processes

A process is stationary (in the strict sense) if

$$p[w;t+\tau] = p[w;t]$$

$$p[w_1, w_2; t_1+\tau, t_2+\tau] = p[w_1, w_2; t_1, t_2]$$

$$\vdots$$

[Stationary process are to random processes as LTI systems are to linear systems.]

For a stationary process, can define the power spectral density

$$S_W(\omega) = \int_{-\infty}^{\infty} \rho(t)\, e^{-j\omega t}\, dt$$

$$= \mathcal{F}[\rho(t)]$$

Conversely,

$$\rho(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, d\omega$$

$$\rho(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega = E[w^2(t)]$$

$$= \text{area under } S(\omega)$$

White Noise

A *white noise* process $w(t)$ is a random process with zero mean and flat power spectrum:

$$S_w(\omega) = W = \text{const.}$$

$$\Rightarrow \rho(\tau) = W\,\delta(\tau)$$

Note that $\rho(0) = \infty \Rightarrow$

$$E[\omega^2(t)] = \infty$$

So the process is not physically possible. But it *is* useful mathematically.

More generally, a non-stationary white noise process satisfies

$$E[\omega(t)] = 0.$$

$$E[\omega(t)\,\omega(\tau)] = \rho(t,\tau) = W(t)\,\delta(t-\tau)$$

So don't have to stick to stationary processes.

Response of a Linear System to White Noise

Consider the system

$$\dot{x} = Ax + Gw, \qquad x(0) = 0$$

where $w(t)$ is a zero-mean, white noise process with

$$E\left[w(t)\,w^T(\tau)\right] = R(t)\,\delta(t-\tau)$$

Statistics of $x(t)$:

$$x(t) = \int_0^t \Phi(t,\tau)\,G(\tau)\,w(\tau)\,d\tau$$

$$E\left[x(t)\right] = E\left[\int_0^t \Phi(t,\tau)\,G(\tau)\,w(\tau)\,d\tau\right]$$

$$= \underline{0}$$

$$E\left[x(t_1)\,x^T(t_2)\right]$$

$$= E\left[\int_0^{t_1} \Phi(t_1,\tau_1)\,G(\tau_1)\,w(\tau_1)\,d\tau_1 \int_0^{t_2} w^T(\tau_2)\,G^T(\tau_2)\,\Phi^T(t_1,\tau_2)\,d\tau_2\right]$$

$$= \int_0^{t_1}\int_0^{t_2} \Phi(t_1,\tau_1)\,G(\tau_1)\,R(\tau_1)\,\delta(\tau_1-\tau_2)\,G^T(\tau_2)\,\Phi^T(t_2,\tau_2)\cdot d\tau_1\,d\tau_2$$

$$P(t_1, t_2) = E\left[x(t_1)\, x^T(t_2)\right]$$

$$= \int_0^{\min(t_1,t_2)} \Phi(t_1,\tau)\, G(\tau)\, R(\tau)\, G^T(\tau)\, \Phi^T(t_2,\tau)\, d\tau$$

Take $t_1 > t_2$. Then

$$P(t_1, t_2) = \int_0^{t_2} \Phi(t_1,\tau)\, G(\tau)\, R(\tau)\, G^T(\tau)\, \Phi^T(t_2,\tau)\, d\tau$$

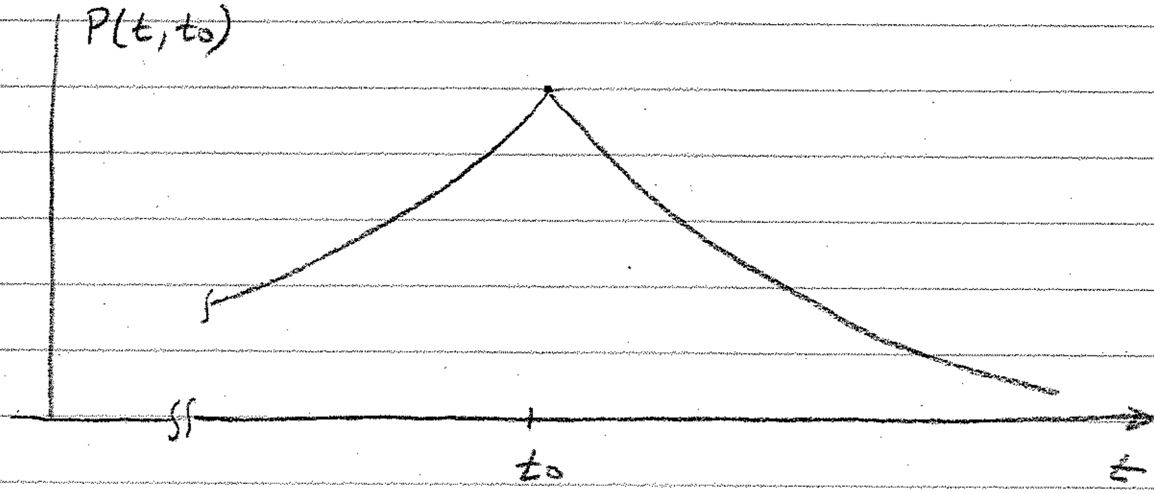
$$= \Phi(t_1, t_2) \int_0^{t_2} \Phi(t_2,\tau)\, G(\tau) \cdots \, d\tau$$

$$= \Phi(t_1, t_2)\, P(t_2, t_2)$$

Likewise, for $t_1 < t_2$,

$$P(t_1, t_2) = P(t_1, t_1)\, \Phi^T(t_2, t_1)$$

Typical (scalar) $P(t, t_0)$



Usually, care about <u>covariance</u> of $x(t)$,

$P(t) \equiv P(t,t)$ Looks like controllability!

$$= \int_0^t \Phi(t,\tau)\, G(\tau)\, R(\tau)\, G^T(\tau)\, \Phi^T(t,\tau)\, d\tau$$

Can easily find d.e. for $P(t)$:

$$\dot{P}(t) = \int_0^t \dot{\Phi}\, G R G^T \Phi^T d\tau + \int_0^t \Phi\, G R G^T \dot{\Phi}^T d\tau$$

$$+ G R G^T$$

$$= \int_0^t A \Phi\, G R G^T \Phi^T d\tau + \int_0^t \Phi\, G R G^T \Phi^T A^T d\tau$$

$$+ G R G^T$$

$\therefore$,

$$\dot{P}(t) = A(t) P(t) + P(t) A^T(t) + G(t) R(t) G^T(t)$$

= "Lyapunov Equation"