SOLUTION OF THE RICCATI EQUATION

x & λ satisfy

$$\begin{Bmatrix} \dot{x} \\ \dot{\lambda} \end{Bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{Bmatrix} x \\ \lambda \end{Bmatrix}$$

$$= H \begin{Bmatrix} x \\ \lambda \end{Bmatrix}$$

$\lambda(t)$ can be expressed in terms of $x(t)$ by

$$\lambda(t) = P(t)\, x(t)$$

If $P(t)$ has reached steady-state, so that $P(t) = P$, then

$$\dot{x} = A\dot{x} - BR^{-1}B^T\lambda = (A - BK)\,x$$

is LTI. So if x is an eigenvector of $A-BK$ (the stable closed-loop dynamics), $\begin{Bmatrix} x \\ Px \end{Bmatrix}$ is an eigenvector of H, corresponding to a stable eigenvalue. So in general, the state and co-state lie in the space

$$\begin{Bmatrix} x \\ \lambda \end{Bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} x , \qquad x \in \mathbb{R}^4$$

But this space is also spanned by $[V]$, the matrix whose columns are the stable eigenvectors of H. Express V as

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

We must have for every x, λ,

$$\begin{Bmatrix} x \\ \lambda \end{Bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \underline{\alpha} , \qquad \underline{\alpha} \in \mathbb{R}^n$$

So $x = V_1 \underline{\alpha} \Rightarrow \underline{\alpha} = V_1^{-1} x$

$\Rightarrow \lambda = V_2 \underline{\alpha} = V_2 V_1^{-1} x$

But $\lambda = Px \; \forall x$. So it must be that

$$\boxed{P = V_2 V_1^{-1}}$$

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad R = 1$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \rightarrow Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow H = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -1 & -3 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

`>> [v, d] = eig(H)` gives eigensystem.
Taking stable part,

$$V = \begin{pmatrix} 0.0307 & -0.6418 \\ -0.0978 & 0.4902 \\ -0.5591 & -0.5111 \\ -0.8227 & 0.2944 \end{pmatrix}$$

and

$$P = V_2 V_1^{2} = \begin{pmatrix} 6.7987 & 7.8577 \\ 7.8577 & 10.8870 \end{pmatrix}$$