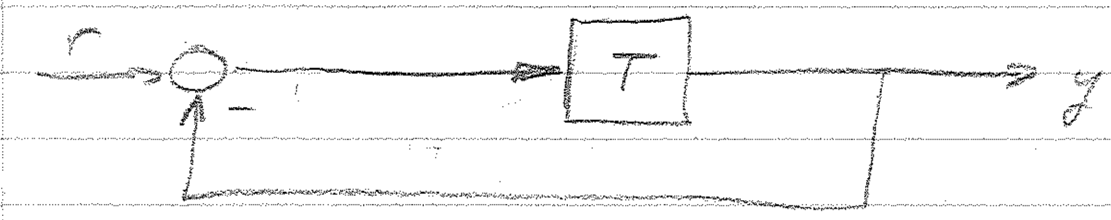
Robustness of SISO systems

Consider a SISO loop

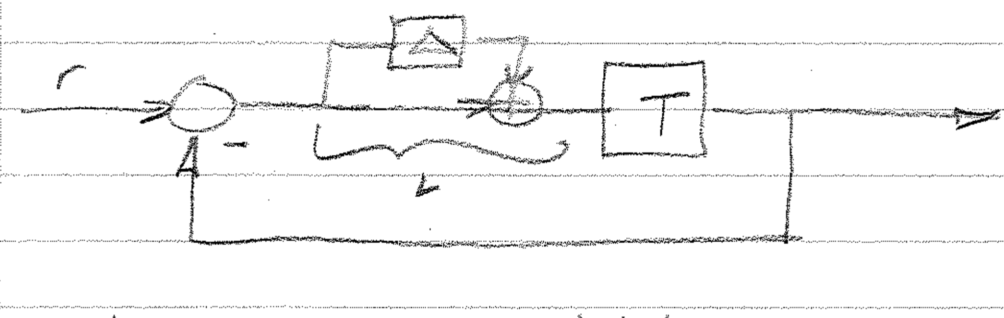


The loop is stable if the Nyquist diagram of $T(j\omega)$ encircles the point -1 N times, counterclockwise, where

$$N = P$$

$\quad$ └ # of o.L unstable poles.

Now consider a modified loop



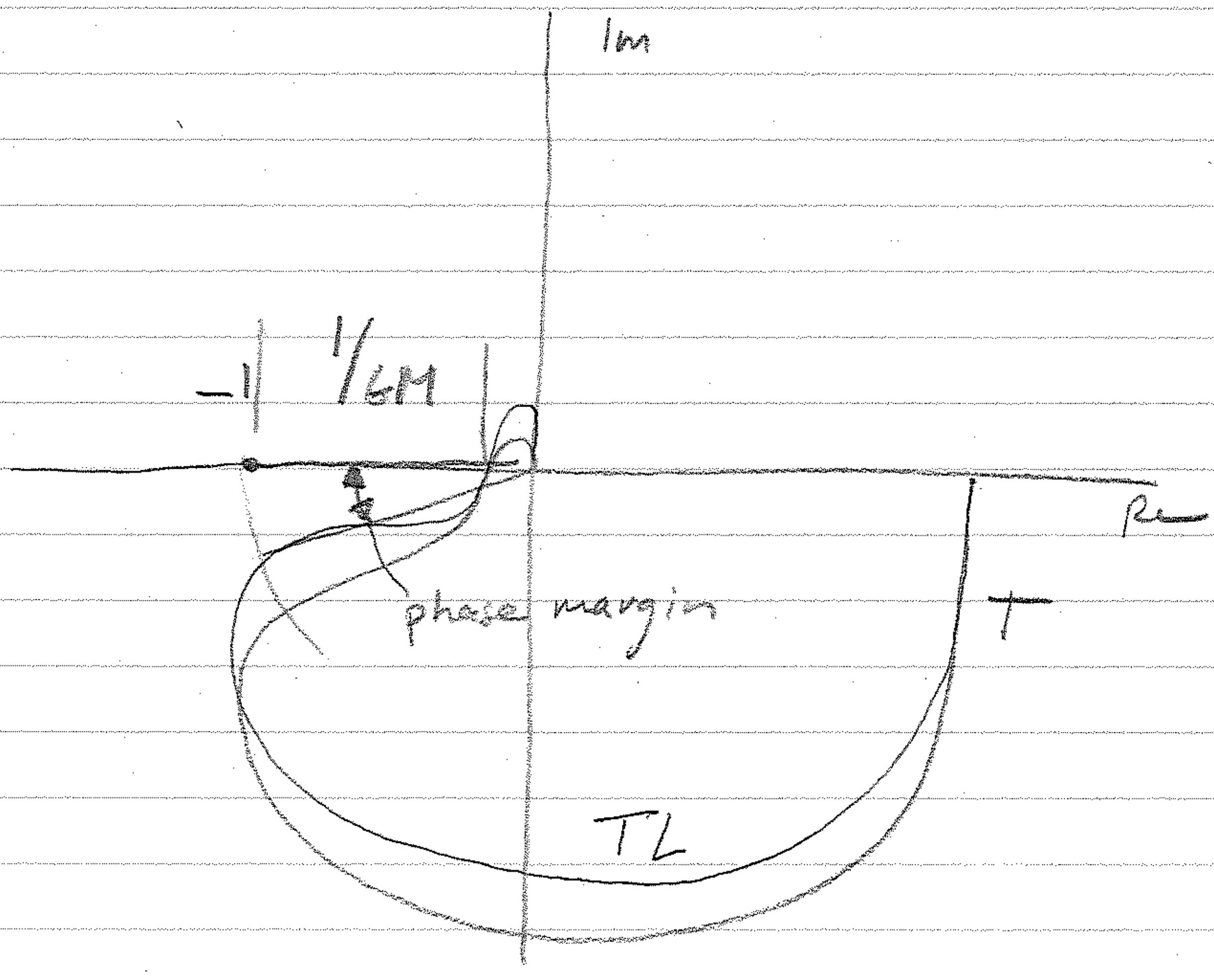
with Δ (L) stable. Is this loop stable?

The loop is stable if the Nyquist plot of $L(j\omega)T(j\omega)$ encircles -1 N times.

Alternatively, the loop is stable in the Nyquist diagram of

$$\Delta \cdot \frac{T}{1+T}$$

encircles -1 0 times.



So gain margin and phase margin are measures of how close T is to -1, hence how much $L \neq 1$ to cause instability. $(\Delta \neq 0)$

If all we know is $|\Delta| < \rho$, then can guarantee stability if

$$\left|\Delta \frac{T}{1+T}\right| = |\Delta| \left|\frac{T}{1+T}\right|$$

$$= \underbrace{\frac{1}{\rho}|\Delta|}_{<1} \ \underbrace{\rho \frac{T}{1+T}}_{\text{want } <1} < 1$$

So can guarantee stability iff

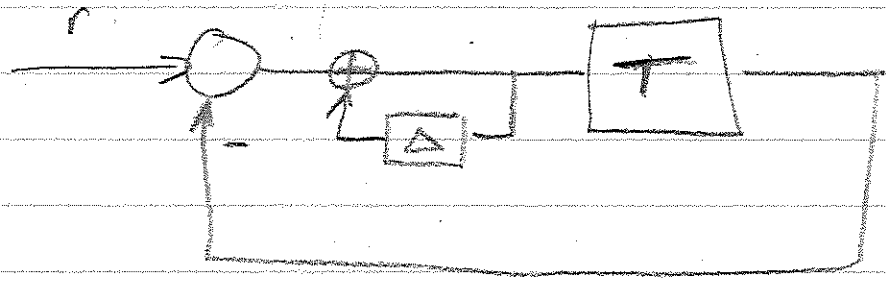
$$\frac{T}{1+T} = \frac{1}{1+1/T} < \frac{1}{\rho}$$

This is a form of the <u>small gain theorem</u>

More generally, need

$$\boxed{|\Delta| < |1+T^{-1}|} \quad \text{(multiplicative)}$$

Division error:



Break loop at Δ:

$$\Rightarrow \text{want } \left|\Delta \frac{1}{1+T}\right| < 1$$

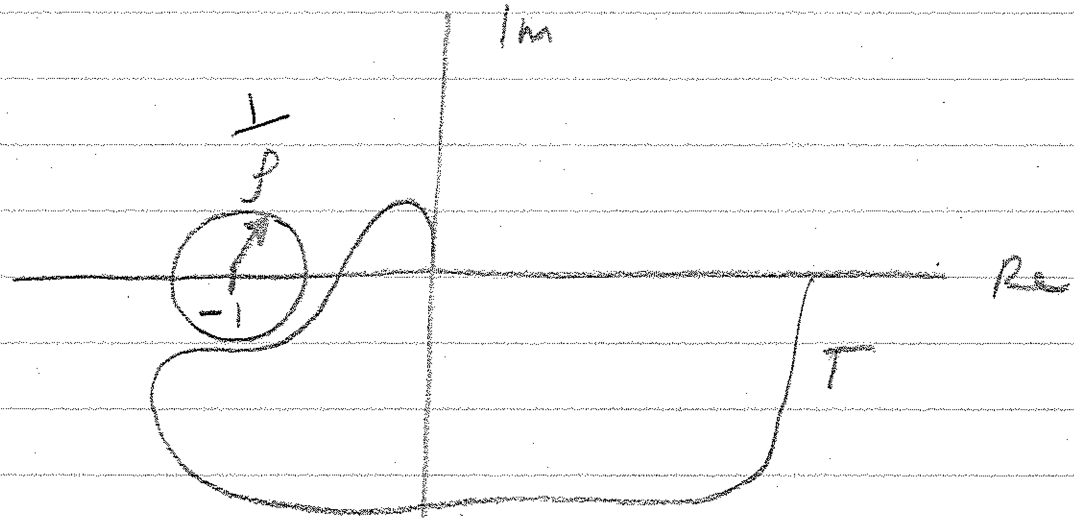
$$\Rightarrow \boxed{|\Delta| < |1+T|} \quad \text{(division)}$$

$1+T$ is the <u>return difference</u>, and plays a central role in control.

So if we know $|\Delta| < \rho$, keep

$$|1 + T| > \frac{1}{\rho}$$

$$\Rightarrow \left|T - (-1)\right| > \frac{1}{\rho}$$



Example LQR is very robust

$$A^T P + PA + E^T E - PB \quad B^T P = 0$$

$$F = \quad B^T P$$

$$\Rightarrow -A^T P - PA = E^T E - F^T F$$

$$\Rightarrow -sP - A^T P - PA + sP = E^T E - F^T F$$

$$(-sI - A^T) P + P(sI - A) =$$

$$\Rightarrow \underset{F}{\cancel{B^T P}}(sI-A)^{-1}B + (\quad)^T$$

$$= B^T(-sI-A^T)^{-1}(E^TE - F^TF)(sI-A)^{-1}B$$

$$F(sI-A)^{-1}B = T$$

$$E(sI-A)^{-1}B = H$$

$$\Rightarrow T(s) + T^T(-s) = H^T(-s)H(s) - T^T(-s)T(s)$$

$$\Rightarrow \left[1+T^T(-s)\right]\left[1+T(s)\right] = 1 + H^T(-s)H(s) \geq 1$$

$$\Rightarrow |1+T(j\omega)| \geq 1 \quad \forall\, \omega$$

So LQR is very robust!

$GM = [1/2, \infty)$!
$PM = 60^\circ$!