The Symmetric Roots Characteristic Equation

We can determine the asymptotic properties of the LQR. Useful for

- insight
- determining how to choose weights.

Can show that optimal state history satisfies

$$\begin{Bmatrix} \dot{x} \\ \dot{\lambda} \end{Bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -E^TE & -A^T \end{bmatrix} \begin{Bmatrix} x \\ \lambda \end{Bmatrix} \quad \begin{matrix} x(0) \text{ given} \\ \lambda(T) = 0 \\ \text{(or other)} \end{matrix}$$

where $J = \int (z^Tz + u^TRu)\,dt$

$z = Ex$ "performance variables"

Poles of "Hamiltonian system" are at

$$\Delta(s) = \begin{vmatrix} sI - A & BR^{-1}B^T \\ E^TE & sI + A^T \end{vmatrix}$$

$$= |sI - A| \left| sI + A^T - E^TE(sI-A)^{-1}BR^{-1}B^T \right|$$

$$\Rightarrow \Delta(s) =$$

$$|sI - A| \cdot |sI + A^T| \cdot$$

$$\left| I - (sI + A^T)^{-1} E^T E (sI - A)^{-1} B R^{-1} B^T \right|$$

$$= \underbrace{|sI - A|}_{\phi(s)} \cdot \underbrace{|-sI - A^T|}_{\phi(-s)} \cdot (-1)^n \cdot$$

$$\left| I + \underbrace{E(sI - A)^{-1} B}_{H(s)} \cdot R^{-1} \cdot \underbrace{B^T(-sI - A^T) E^T)}_{H^T(-s)} \right|$$

$$= (-1)^n \phi(s) \phi(-s) \left| I + H(s) R^{-1} H^T(-s) \right|$$

If we let $R = \rho \bar{R}$, then

$$\Delta(s) = (-1)^n \phi(s) \phi(-s) \left| I + \frac{1}{\rho} H(s) \bar{R} H^T(-s) \right|$$

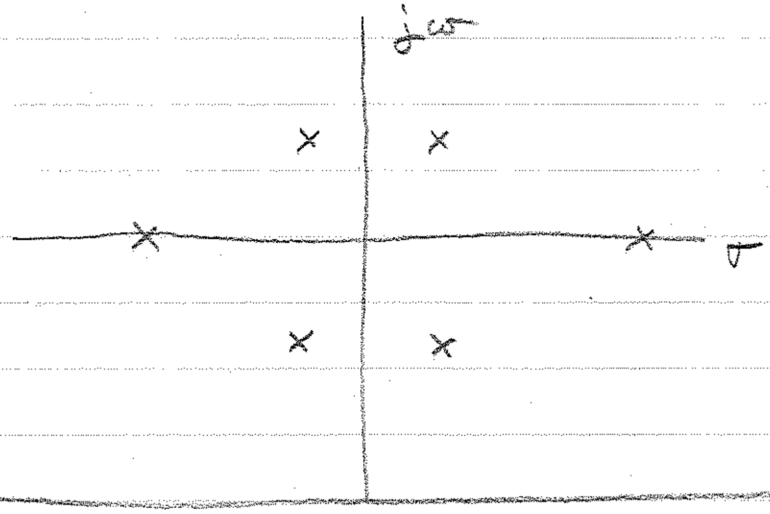
This is the <u>symmetric roots</u> <u>characteristic equation</u>.

Note that

$$\Delta(s) = 0 \Rightarrow \Delta(-s) = 0$$

$\Rightarrow$ If s is a root, then so is $-s$.

Thus, the "poles" of the system would look like:



The stable poles are the poles of $A - BF$, the optimal steady-state regulator.

In SISO case, $H(s)$ is a scalar,

$$H(s) = \frac{\psi(s)}{\phi(s)}$$

So $$\Delta(s) = (-1)^n \phi(s)\,\phi(-s)\left(1 + \frac{1}{\rho}\,\frac{\psi(s)\,\psi(-s)}{\phi(s)\,\phi(-s)}\right)$$

$$\Delta(s) = 0 \Rightarrow \phi(s)\,\phi(-s) + \frac{1}{\rho}\,\psi(s)\,\psi(-s) = 0$$

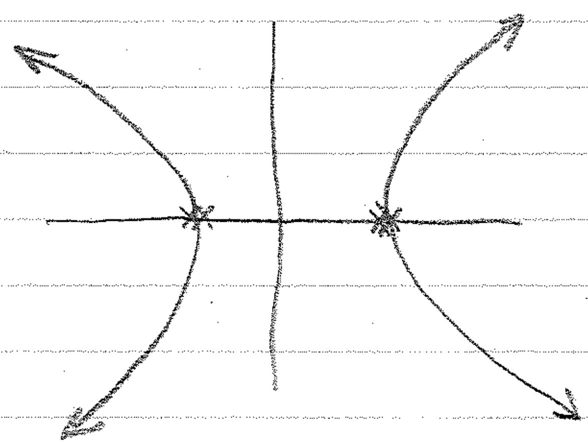
This equation looks like a root locus!

- "poles" are poles of A, plus their reflection about $j\omega$
- "zeros" are zeros of transfer function from control to performance, plus their reflection
- Closed-loop poles are resulting stable poles

<u>Example</u> Inverted pendulum

$$H(s) = \frac{1}{s^2 - \Omega^2} = \frac{1}{(s-\Omega)(s+\Omega)}$$

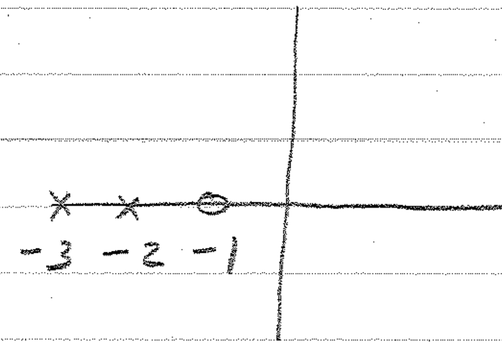
$$\Delta(s) = 0 \Rightarrow (s-\Omega)(s+\Omega)(-s-\Omega)(-s+\Omega) + \frac{1}{\rho} = 0.$$



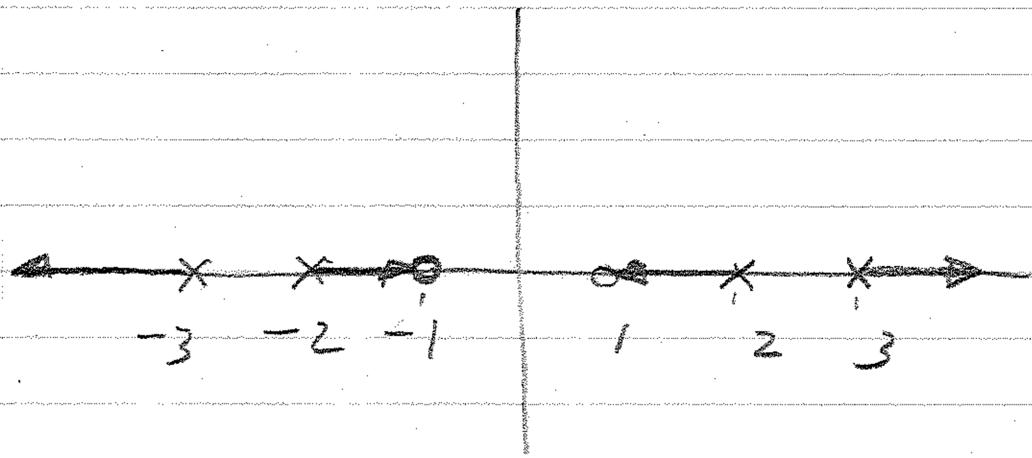
as before!

Example

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$



Root locus:



Expensive control ($\rho \to \infty$):

$s = -2, -3$ do nothing!

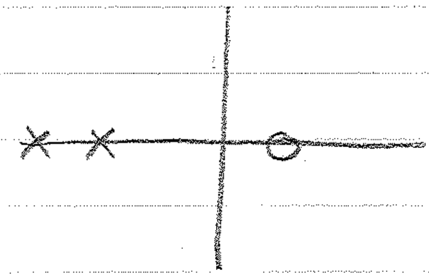
Cheap control: ($\rho \to 0$)

$s \approx -1$ "hide" pole

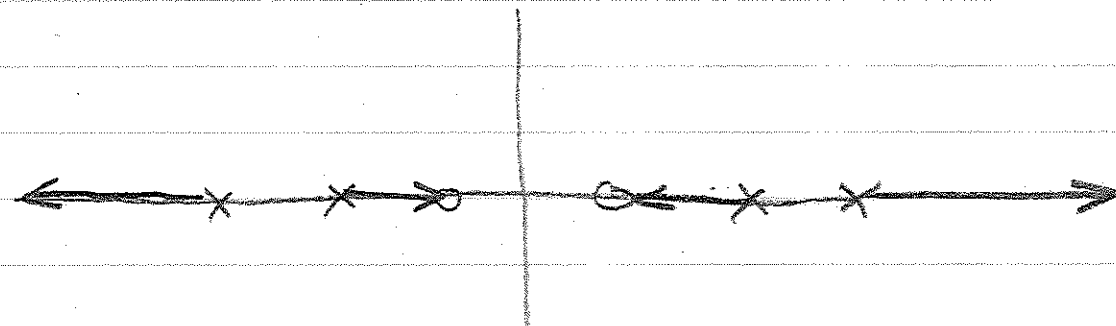
$s \approx -\frac{1}{\sqrt{\rho}}$ make response very fast.

Example

$$H(s) = \frac{s-1}{(s+1)(s+3)}$$



Root locus:



(The same!)

Expensive control ($\rho \to \infty$):

$s = -2, -3$ do nothing!

Cheap control:

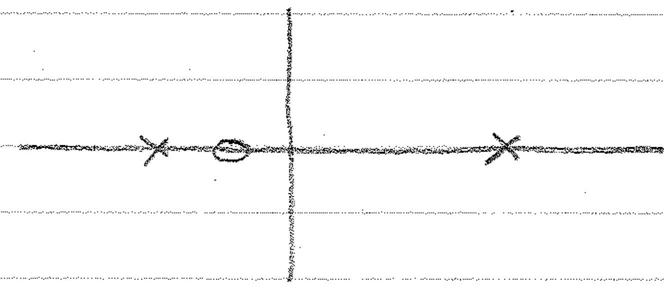
$s \approx -1/\sqrt{\rho}$ make response fast

$s \approx -1$ make "wrong way" behavior as small as possible

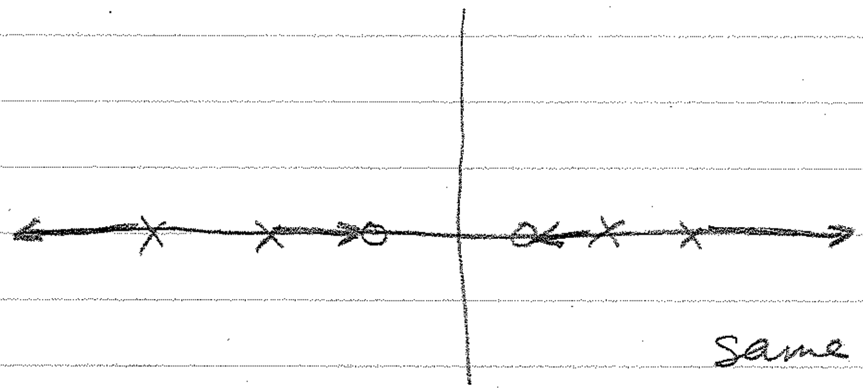
(not hidden)

Example

$$H(s) = \frac{s+1}{(s-2)(s+3)}$$



Root locus:



Expensive control: $(\rho \to \infty)$

$s = -3$ leave stable mode alone

$s = -2$ stabilize unstable mode with minimum effort

Cheap control: $(\rho \to 0)$

$s = -1/\sqrt{\rho}$ fast response

$s \approx -1$ "hide" one mode.

In general,

expensive control:

- poles stay put or move to their stable reflection

cheap control:

- Poles go to m zeros (or their stable reflections)

- n-m poles go to butterworth pattern

$$|s| \sim \frac{1}{(\rho^{1/2})^{1/n-m}}$$