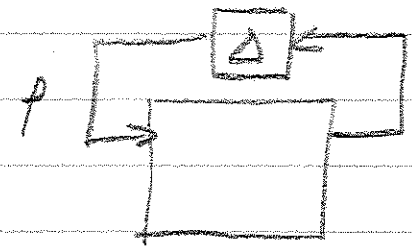
Structured Uncertainties

So far, have look at unstructured uncertainties:

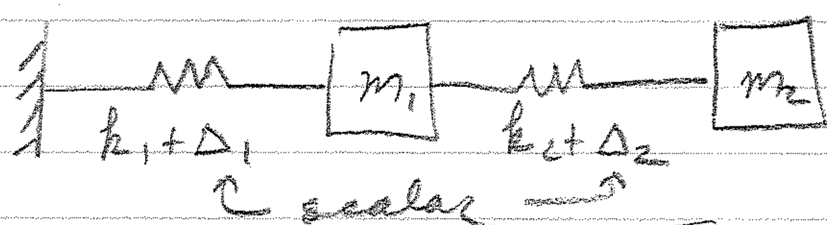


p, q vectors Δ matrix

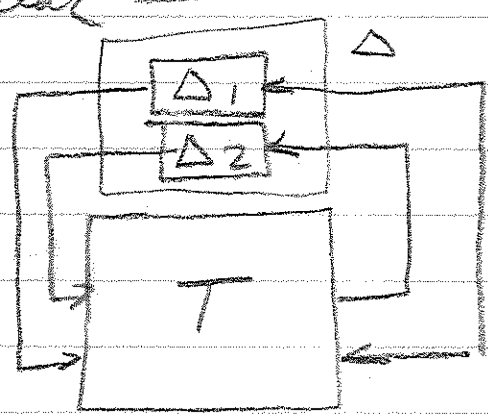
$$\|\Delta\|_\infty < 1 \quad (\text{or } \bar{\sigma}(\Delta(j\omega)) < 1)$$

In practice, the Δ matrix may be structured

Example



model:



so $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$

Suppose we know $\bar{\sigma}(\Delta_1) \leq 1$, $\bar{\sigma}(\Delta_2) \leq 1$. Define

$$\mathcal{D} = \left\{ \Delta : \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix},\ \bar{\sigma}(\Delta_1) \leq 1, \bar{\sigma}(\Delta_2) \leq 1 \right\}$$

I.e., allowable Δ's are

$$\Delta \in \mathcal{D}$$

Can we use earlier methods to determine stability of the loop?

Sort of.

Note that

$$\Delta \in \mathcal{D} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \bar{\sigma}(\Delta) \leq 1$$

Example $\quad \Delta = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$

$$\Rightarrow \bar{\sigma}(\Delta) = 0.2$$

But $\quad \Delta \notin \mathcal{D}$!

So system is stable if

$$\bar{\sigma}(T) < 1$$

But this is sufficient, not necessary

Example

$$T = \begin{bmatrix} 1/2 & 100 \\ 0 & 1/2 \end{bmatrix}$$

$$\bar{\sigma}(T) = 100.025$$

But $_\wedge$ stable for all $\Delta \in \mathcal{D}$

$(I - \Delta T)^{-1}$

The structured singular value

What we need is a way to describe how big T is, that takes into account the structure of the uncertainty.

Define the function

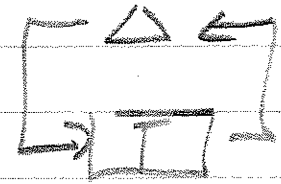
$$\mu(T) = \frac{1}{k},$$

where k is the largest k such that

$$\det(I + k\Delta T) \neq 0 \text{ for all } \Delta \in \mathcal{D}$$

$\mu(T)$ is the structured singular value of T.

Theorem. The loop



is stable for all $\Delta \in \mathcal{D}$ iff $\mu(T) < 1$. $\forall \omega$.

Hard to calculate $\mu(T)$, but can bound, and bounds allow for control design.

Uncertainty blocks:

Repeated real scalars:

$$\Delta_i = \delta_i^r I$$

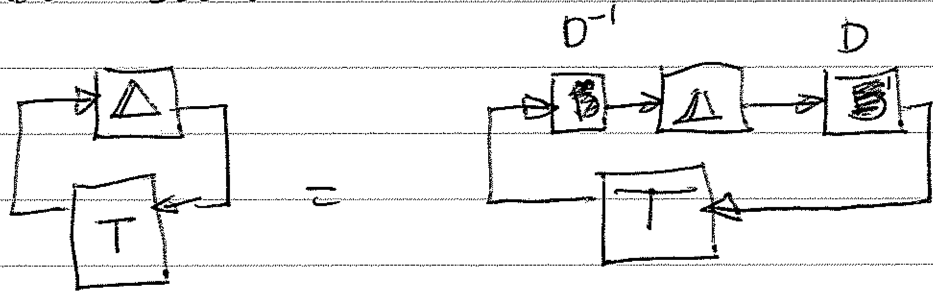
Repeated complex scalars:

$$\Delta_i = \delta_i^c I$$

Complex matrix

$$\Delta_i = \text{complex matrix}$$

Basic idea for computing μ: If D_i commutes with Δ_i, then can redraw block



where $\quad D = \text{diag}(D_1, D_2, D_3 \ldots)$

Note that since $\mu(T) \le \bar{\sigma}(T)$, it is also true that

$$\mu(T) \le \bar{\sigma}(D T D^{-1})$$

So can optimize over D to find best (smallest) upper bound.

For $\quad \Delta_i = \delta_i^c I, \; D_i = \text{any matrix}$

For $\quad \Delta_i = \text{complex matrix}, \; D_i = d_i I$

Will come back to real case.

Note that

$$\bar{\sigma}(D T D^{-1}) = \alpha$$

$$\Rightarrow p^* D^{-*} T^* D^* D T \underbrace{D^{-1} p}_{\bar{p}} \le \alpha \, p^* p$$

$$\Rightarrow \cancel{\bar{p}^* T^* D^* D T \bar{p}}$$

$$\bar{p}^* \cancel{D^{-*}} T^* D^* D T \bar{p} \le \alpha \, \bar{p}^* \underbrace{D^* D}_{\bar{D}} \bar{p}$$

$$\Rightarrow \quad \bar{p}^* T^* \bar{D} T \bar{p} \le \alpha \, \bar{p}^* \bar{D} \bar{p}$$

$$\Longrightarrow \quad T^* \bar{D} T \leq \alpha \bar{D}$$

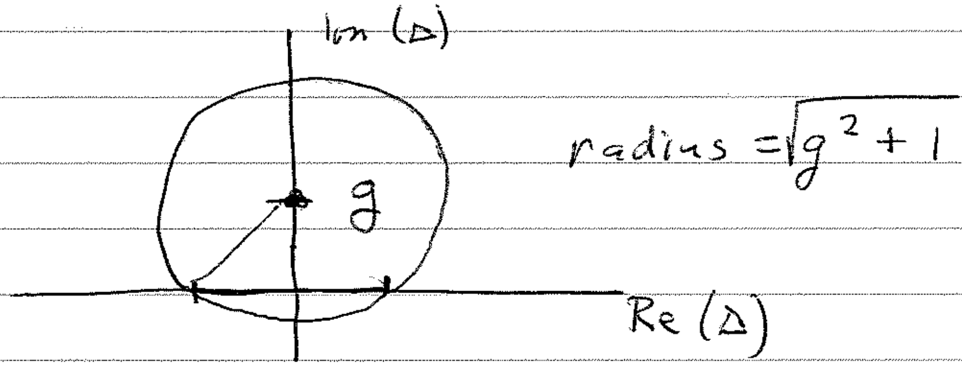
This is a linear matrix inequality (LMI), which is relatively easily solved. So can do a "γ-iteration" to find (bound on) μ, which also generates $\bar{D}$.

With real uncertainties, can add term:

$$T^* \bar{D} T + j(GT - T^* G) \leq \alpha \bar{D}$$

Note that $\bar{D} = \bar{D}^* > 0$, $G = G^*$. G must also commute appropriately with the real Δ blocks. ($G = 0$ for complex blocks.)

G term corresponds to covering a real Δ with off-axis circle:



D–K iteration:

1. For D (and G) given, find controller K(s) that minimizes ∞-norm of $D T_{zw} D^{-1}$. This requires H_∞ controller and γ iteration.

2. For given K(s), find D (and G) that minimizes $\bar{\sigma}(D T_{zw} D^{-1})$. This can be done with efficient LMI techniques. Also requires γ-iteration.

3. Repeat until desired convergence.