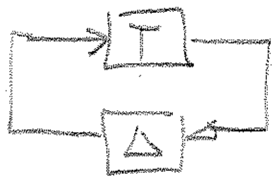
The Infinity Norm: (supremum, really)

$$\|\Delta(s)\|_\infty \equiv \max_{\omega} \bar{\sigma}(\Delta(j\omega))$$

<u>The Small Gain Theorem</u> (again)

The loop



is stable for all $\|\Delta\|_\infty \le \rho$ iff $\|T\|_\infty < 1/\rho$.

Testing for $\|T\|_\infty < \gamma$

Suppose I want to test for $\|T\|_\infty < \gamma$. How can I do it?

Algorithm 1 (Poor)

1. Express $T(s)$ in state space form:

$$\dot{x} = Ax + B_p p$$
$$q = C_q x$$

2. Over the frequency range of interest, pick N test frequencies, ω_i.

3. For each ω_i, calculate

$$T(j\omega_i) = T_i = C_q(j\omega_i I - A)B_p$$

4. Calculate $T_i^* T_i$

5. Use an eigenvalue solver to find $\bar{\sigma}(T_i^* T_i) = \sigma_i$

6. $\|T\|_\infty \approx \max_i \sigma_i$

Problems

1. Requires a lot of calculation — N may be 100 — 1000 or more.
2. It is possible to miss an important frequency.
3. inelegant.

There is a much better algorithm.

Argument if $\|T\|_\infty \geq \gamma$, then there is a frequency ω such that

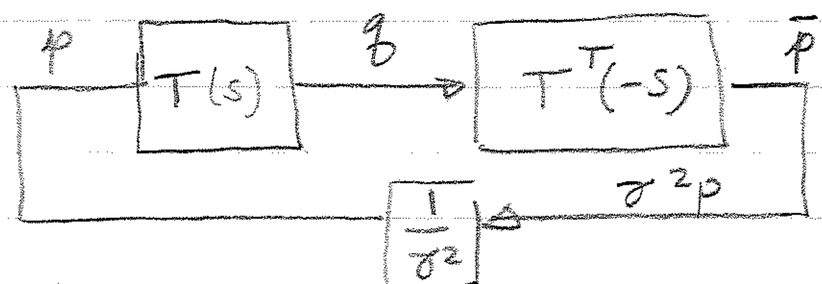
$$\bar{\sigma}\left[T(j\omega)\right] = \gamma$$

(Since $T(j\omega) \to 0$ as $\omega \to \infty$, and $\bar{\sigma}(T(j\omega))$ is continuous).

At that frequency, There is a p (complex), such that

$$T^*(j\omega)\,T(j\omega)\,p = \gamma^2 p$$

So we can close the following loop



and it will oscillate at frequency ω.
$(s = j\omega)$.

So, see if loop has any $j\omega$-axis poles — if it does, $\|T\|_\infty \geq \gamma$.

$$\left.\begin{aligned} \dot{x} &= Ax + B_p p \\ q &= C_q x \end{aligned}\right\} T(s)$$

$$\left.\begin{aligned} \dot{x}_2 &= -A^T x_2 - C_q^T x_2 \\ \bar{p} &= B_p^T x_2 \end{aligned}\right\} T^T(-s)$$

$$p = \frac{1}{\gamma^2}\bar{p} = \frac{1}{\gamma^2} B_p^T x_2$$

Put into single state space equation:

$$\begin{pmatrix} \dot{x} \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} A & \frac{1}{\gamma^2} B_p B_p^T \\ -C_q^T C_q & -A^T \end{bmatrix}}_{\text{Hamiltonian!}} \begin{pmatrix} \dot{x} \\ x_2 \end{pmatrix}$$

We have a Hamiltonian, just like in LQR case, <u>except</u>:

1) $Q = C_q^T C_q$; $R = \gamma^2 I$
2) Sign of BB^T term $+$, not $-$.

Corresponding Ricatti equation:

$$0 = A^T P + PA + C_q^T C_q \underline{\underline{+}} \frac{1}{\gamma^2} P B_p B_p^T P$$

Significance of + sign:

$$J = \frac{1}{2} \int_0^\infty \left[|q|^2 - \gamma^2 |p|^2 \right] dt$$

If Hamiltonian oscillates, cost is not finite $\Longrightarrow$ can make

$$\|q\|_2^2 > \gamma^2 \|p\|_2^2$$

$$\Longrightarrow \quad \frac{\|q\|_2}{\|p\|_2} > \gamma$$

$$\Longrightarrow \quad \|T\|_\infty > \gamma$$

<u>Neat!</u>

Algorithm 2 (good)

To test if T stable, and $\|T\|_\infty < \gamma$,

1. Form the Hamiltonian

$$H = \begin{bmatrix} A & \frac{1}{\gamma^2} B_p B_p^T \\ -C_q^T C_q & -A^T \end{bmatrix}$$

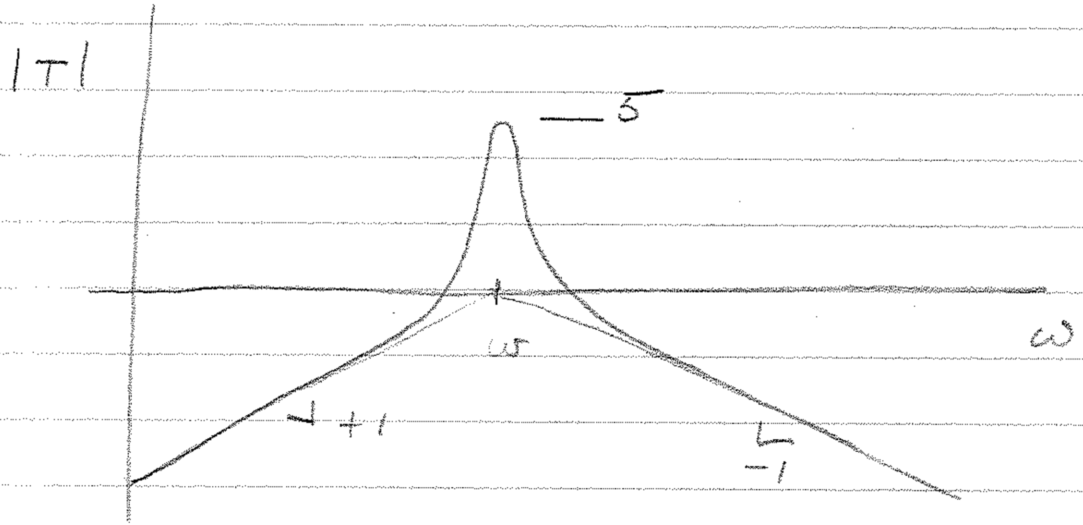
2. Find the eigenvalues; if any are on $j\omega$ axis, $\|T\|_\infty \not< \gamma$

3. If none on $j\omega$ axis, solve for Riccati matrix P. If $P > 0$, T stable, and $\|T\|_\infty < \gamma$.

Note: Can find $\|T\|_\infty$ by doing a bisection search on γ.

Example Find the ∞-norm of

$$T(s) = \frac{s}{s^2 + 0.2s + 1}$$



So $\|T\|_\infty = 5$.

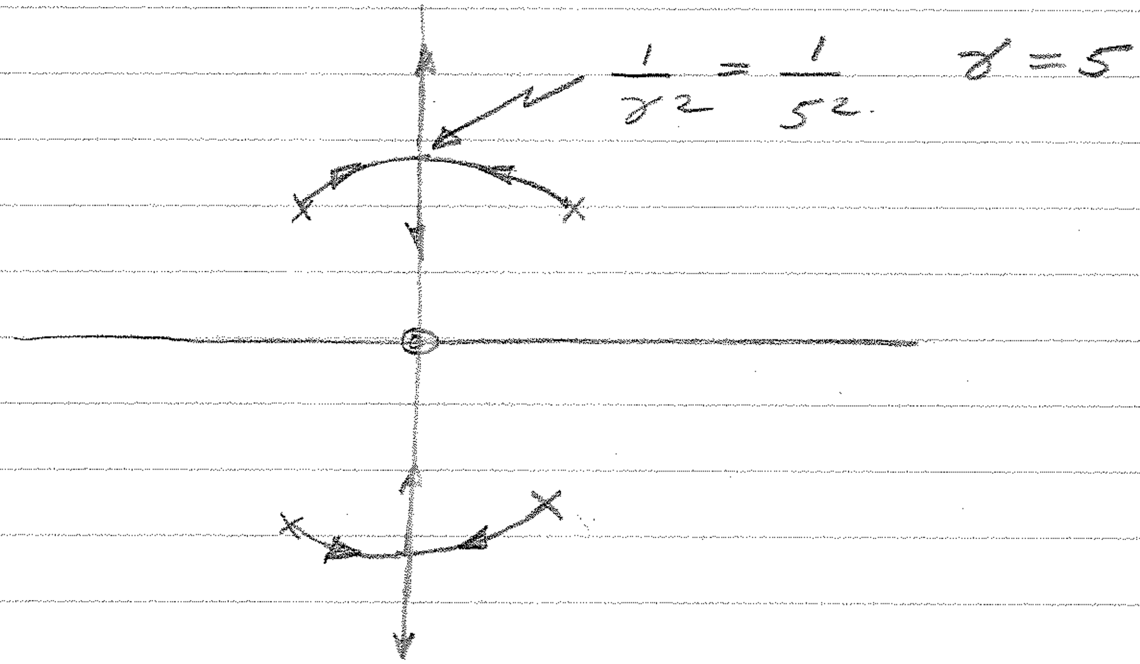
Alternatively,

$$T(-s)T(s) = \frac{-s}{s^2 - 0.2s + 1} \; \frac{s}{s^2 + 0.2s + 1}$$

Want to know when $T(-j\omega)T(j\omega) = \gamma^2$

$$\Rightarrow \quad 1 - \frac{1}{\gamma^2} T(-s)T(s) = 0 \quad \text{on } j\omega\text{-axis}$$

So draw locus as function of $\frac{1}{\gamma^2}$:



Looks like SRCE, but with poles on $j\omega$ axis!!