Problem 1: Plot the root locus diagram for positive values of $K$ for the solutions of the equation

$$s^3 + (5 + K)s^2 + (6 + K)s + 2K = 0$$

Solution: The equation can be rewritten as

$$s^3 + 5s^2 + 6s + K(s^2 + s + 2) = 0$$

This equation is essentially the characteristic equation for a system with open-loop zeros at the roots of $s^2 + s + 2 = 0$ and open-loop poles at the roots of $s^3 + 5s^2 + 6s = 0$. So the zeros are at

$$z_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2} \approx -0.5 \pm 1.3229j$$

The poles are at

$$p_1 = 0$$
$$p_2 = -3$$
$$p_3 = -2$$

The locus for positive $K$ must include the region on the real line $-2 < s < 0$ and $s < -3$, since these regions are to the left of an odd number of poles and zeros. There is $3 - 2 = 1$ asymptote, which is that $s \to -\infty$ as $K \to \infty$.

The Matlab root locus may be obtained by the commands

```matlab
>> num = [1 1 2];
>> den = [1 5 6 0];
>> rlocus(num,den)
>> title('')
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = ylabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> h = gca;
>> set(h,'fontsize',14);
>> print -depsc 'figure1.eps'
```

The first three commands generate the root locus. The last six commands clean up the plot a bit and produce a plot file for this solution. The Matlab plot is shown below:
Problem 2: The open loop transfer function of a closed-loop control system with unity negative gain feedback is

\[ G(s) = \frac{K}{s(s + 3)(s^2 + 6s + 64)} \]

Plot the root locus for this system, and then determine the closed-loop gain that gives an effective damping ratio of 0.707.

Solution: The root locus may be obtained by the commands:

```matlab
>> den = conv([1 3 0],[1 6 64])
den =
    1     9    82   192   0
>> num = [0 0 0 0 1];
>>
>> % set the range of gains fine enough to figure out the right gain
>> % to get 0.707 damping
>>
>> k=logspace(-5,5,10000);
>>
>> % Do the root locus
>>
>> rlocus(num,den,k)
```

At this point, I used the cursor tool in the plot window to find that \( K = 232 \) gives the desired damping ratio. Then I sent the plot to a file:

```matlab
>> title('')
```
The resulting root locus plot is

Does the closed-loop system have the desired behavior, that is, does it behave like a pure second order system with damping ratio $\zeta = 0.0707$? To see, form the closed-loop transfer function

$$ H_{CL}(s) = \frac{G(s)}{1 + G(s)} $$

and plot its step response, $y_1(t)$, compared to the step response $y_2(t)$ of the ideal system

$$ H_{0.707} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} $$

where $\zeta = 0.707$, and $\omega_0 = 1.95$, again using Matlab:

```matlab
>> % Gain that gives 0.707 damping to dominant poles
>> K = 232;
>> % Time vector for plot
>> t = 0:.01:10;
```
% Closed-loop numerator
numcl = K*num
numcl =
    0   0   0   0  232

% Closed-loop denominator
dencl = den +K*num;
dencl =
    1   9  82  192  232

% Closed-loop step response
y1 = step(numcl,dencl,t);

% Ideal second order response
zeta = 0.707;
w0 = 1.95;
num2 = [0 0 w0^2];
den2 = [1 2*zeta*w0 w0^2];
y2 = step(num2,den2,t);

% Do the plot
plot(t,y1,'b',t,y2,'r')
h = gca;
set(h,'fontsize',14);
h = xlabel('Time, \{t\}, (s)')
set(h,'fontsize',14);
h = ylabel('Step Response')
set(h,'fontsize',14);

print -depsc 'figure3.eps'

The result is shown in the plot below:
Note that \( y_1 \) and \( y_2 \) are quite close. The major difference is that the step response of the closed-loop system is flatter near \( t = 0 \), which is a result of the fourth order dynamics of the system.

**Problem 3:** A unity gain negative feedback system has an open-loop transfer function given by

\[
G(s) = \frac{K(1 + 5s)}{s(1 + 10s)(1 + s)^2}
\]

Draw a Bode diagram for this system and determine the loop gain \( K \) required for a phase margin of 20 deg. What is the gain margin?

A lag compensator

\[
G_c(s) = \frac{1 + 10s}{1 + 50s}
\]

is added to this system. Use Bode diagrams to find the reduction in steady state error following a ramp change to the reference input, assuming that the 20 deg phase margin is maintained.

**Solution:** Using Matlab, plot the Bode plot:

```matlab
>> num = [0 0 0 5 1];
>> den = conv([10 1 0],[1 2 1])
den =
    10    21    12    1    0
>> bode(num,den)
>> w = logspace(-3,2,3000);
>> bode(num,den,w)
```

The Bode plot is shown below:
Using the cursor tool, I found that the phase is -160 deg (20 deg phase margin) at \( \omega = 0.59 \). Find the magnitude at that frequency:

\[
>> [\text{mag,phase}] = \text{bode}(\text{num,den},0.59)
\]

\[
\text{mag} = 0.6544
\]

\[
\text{phase} = -160.1873
\]

So to get crossover at that frequency, must choose

\[
K = \frac{1}{0.6544} = 1.528
\]

Next, use the cursor tool to find the frequency at which the phase is -180 deg, to find the gain margin. The phase is -180 deg at \( \omega = 0.897 \), at which frequency the gain of the original system is 0.3145:

\[
>> [\text{mag,phase}] = \text{bode}(\text{num,den},0.897)
\]

\[
\text{mag} = 0.3145
\]

\[
\text{phase} = -179.9924
\]

So the gain of the system with the new gain \( K \) included at \( \omega = 0.897 \text{r/s} \) is \( 1.528 \times 0.3145 = 0.481 \). The gain margin is the reciprocal of this number, so that

\[
\text{Gain margin} = 2.08
\]
Now add the lag compensator

\[ G_c(s) = \frac{1 + 10s}{1 + 50s} \]

The new loop transfer function numerator and denominator (again assuming \( K = 1 \)) is found in Matlab via

```matlab
>> num2 = conv(num,[10 1])
num2 =
  0 0 0 50 15 1
>> den2 = conv(den,[50 1])
den2 =
  500 1060 621 62 1 0
```

The bode plot is below:

![Bode Diagram](image)

Plotting the Bode plot and using the cursor tool yields a crossover frequency of \( \omega_c = 453 \text{ r/s} \) to achieve a phase margin of 20 deg. At this frequency, the magnitude of the transfer function with gain \( K = 1 \) is 0.2, so we need \( K = 5 \) to get the desired crossover frequency.

For the original control system with \( K = 1.528 \) and no lag compensator, the velocity constant is

\[ K_v = \lim_{s \to 0} sG(s) = K = 1.528 \]

For the modified system with \( K = 5 \) and the lag compensator, the velocity constant is

\[ K_v = \lim_{s \to 0} sG(s)G_c(s) = K = 5 \]

Therefore, the steady-state error to a unit ramp is reduced from \( 1/K_v = 0.654 \) for the original control system, to \( 1/K_v = 0.2 \) for the modified control system, a reduction a factor of 3.27.
Problem 4: Plot the Nyquist diagram for the plant with the unstable open-loop transfer function

\[ G(s) = \frac{K(s + 0.4)}{s(s^2 + 2s - 1)} \]

Determine the range of \( K \) for which the closed-loop system with unity negative gain feedback which incorporated this plant would be stable.

Solution: This system has one unstable pole (at \( s = 0.4142 \)), an we need therefore one counter clockwise encirclement of the point \(-1/K\) for stability. Use Matlab to plot the Nyquist diagram:

```matlab
>> num = [1 0.4];
>> den = [1 2 -1 0];
>> nyquist(num,den,.1:.01:10);axis equal
>> title("
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = xlabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> h = gca;
>> set(h,'fontsize',14);
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = ylabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> print -depsc 'figure6.eps'
```

The nyquist plot is below:
The closure of the Nyquist plot is a bit tricky. Since the Nyquist contour in the $s$-plane must detour around the pole at $s = 0$ in a counterclockwise direction, the Nyquist plot above closes in a clockwise direction. So to get one CCW encirclement, $-1/K$ must be inside the small loop near the origin. Blowing up the plot shows that the real line inside the loop is

$$-0.8 < -1/K < 0$$

which implies that

$$K > \frac{1}{0.8} = 1.25$$

Indeed, the characteristic equation for $K = 1.25$ is

$$\phi(s) = (s^3 + 2s^2 - s) + 1.25 (s + 0.4) = s^3 + 2s^2 + 0.25s + 0.5 = 0$$

which has roots at

$$s = -2, \quad s = \pm 0.5j$$

That is, for $K = 1.25$, the closed-loop system is at the boundary of stability, with oscillatory poles on the $j\omega$-axis, as is predicted by Nyquist.