16.31 Homework 1 — Solution

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Problem 1: Plot the root locus diagram for positive values of K for the solutions of the equation

$$s^3 + (5 + K)s^2 + (6 + K)s + 2K = 0$$

Solution: The equation can be rewritten as

$$s^3 + 5s^2 + 6s + K\left(s^2 + s + 2\right) = 0$$

This equation is essentially the characteristic equation for a system with open-loop zeros at the roots of $s^2 + s + 2 = 0$ and open-loop poles at the roots of $s^3 + 5s^2 + 6s = 0$. So the zeros are at

$$z_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2} \approx -0.5 \pm 1.3229j$$

The poles are at

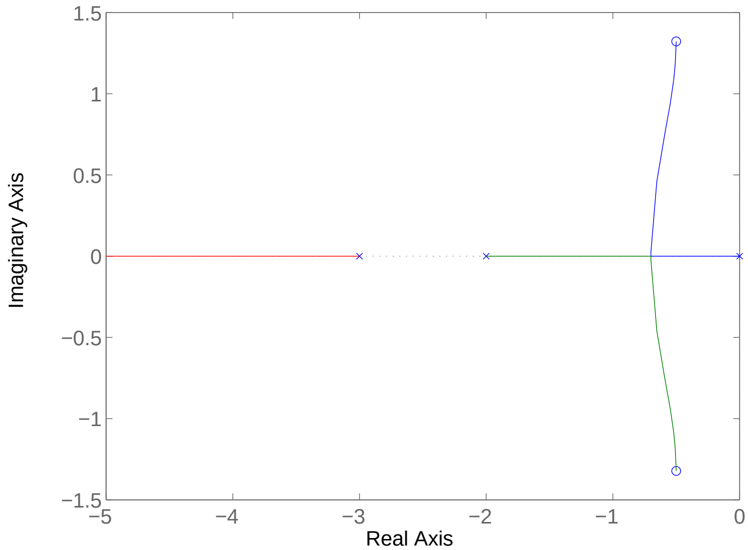
$$\begin{aligned} p_1 &= 0 \\ p_2 &= -3 \\ p_3 &= -2 \end{aligned}$$

The locus for positive K must include the region on the real line $-2 < s < 0$ and $s < -3$, since these regions are to the left of an odd number of poles and zeros. There is $3 - 2 = 1$ asymptote, which is that $s \to -\infty$ as $K \to \infty$.

The Matlab root locus may be obtained by the commands

```
>> num = [1 1 2];
>> den = [1 5 6 0];
>> rlocus(num,den)
>>
>> title('')
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = ylabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> h = gca;
>> set(h,'fontsize',14);
>> print -depsc 'figure1.eps'
```

The first three commands generate the root locus. The last six commands clean up the plot a bit and produce a plot file for this solution. The Matlab plot is shown below:



Problem 2: The open loop transfer function of a closed-loop control system with unity negative gain feedback is

$$G(s) = \frac{K}{s(s+3)(s^2+6s+64)}$$

Plot the root locus for this system, and then determine the closed-loop gain that gives an effective damping ratio of 0.707.

Solution: The root locus may be obtained by the commands:

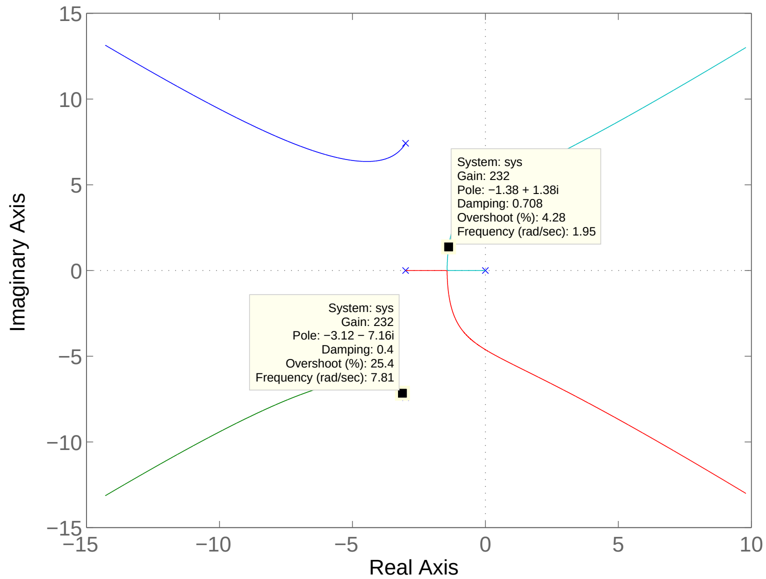
```
>> den = conv([1 3 0],[1 6 64])
den =
     1     9    82   192     0
>> num = [0 0 0 0 1];
>>
>> % set the range of gains fine enough to figure out the right gain
>> % to get 0.707 damping
>>
>> k=logspace(-5,5,10000);
>>
>> % Do the root locus
>>
>> rlocus(num,den,k)
```

At this point, I used the cursor tool in the plot window to find that $K = 232$ gives the desired damping ratio. Then I sent the plot to a file:

```
>> title('')
```

```
>> h = gca;
>> set(h,'fontsize',14);
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = ylabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> print -depsc 'figure2.eps'
```

The resulting root locus plot is



Does the closed-loop system have the desired behavior, that is, does it behave like a pure second order system with damping ratio $\zeta = 0.0707$? To see, form the closed-loop transfer function

$$H_{CL}(s) = \frac{G(s)}{1 + G(s)}$$

and plot its step response, $y_1(t)$, compared to the step response $y_2(t)$ of the ideal system

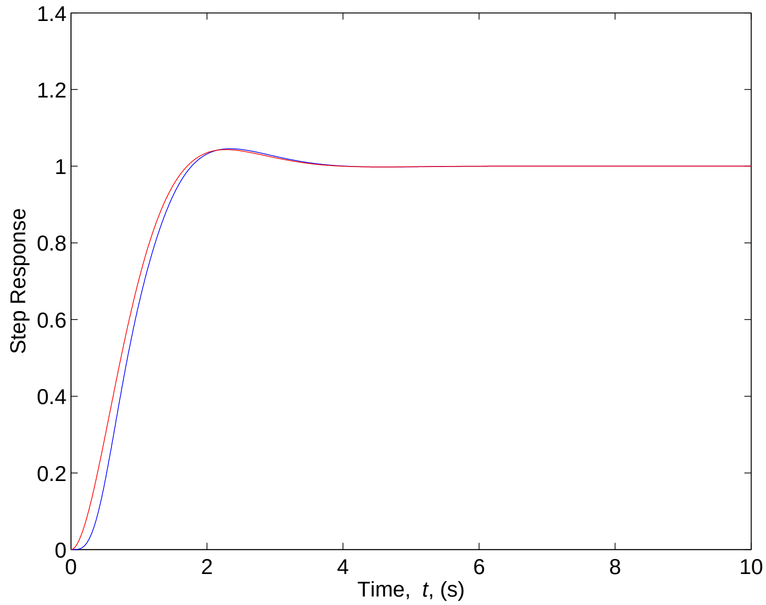
$$H_{0.707} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where $\zeta = 0.707$, and $\omega_0 = 1.95$, again using Matlab:

```
>> % Gain that gives 0.707 damping to dominant poles
>>
>> K = 232;
>>
>> % Time vector for plot
>>
>> t = 0:.01:10;
```

```
>>
>> % Closed-loop numerator
>>
>> numcl = K*num
numcl =
     0     0     0     0   232
>>
>> % Closed-loop denominator
>>
>> dencl = den +K*num;
dencl =
     1     9    82   192   232
>>
>> % Closed-loop step response
>>
>> y1 = step(numcl,dencl,t);
>>
>> % Ideal second order response
>>
>> zeta = 0.707;
>> w0 = 1.95;
>> num2 = [0 0 w0^2];
>> den2 = [1 2*zeta*w0 w0^2];
>> y2 = step(num2,den2,t);
>>
>> % Do the plot
>>
>> plot(t,y1,'b',t,y2,'r')
>> h = gca;
>> set(h,'fontsize',14);
>> h = xlabel('Time, {\it t}, (s)')
>> set(h,'fontsize',14);
>> h = ylabel('Step Response')
>> set(h,'fontsize',14);
>>
>> print -depsc 'figure3.eps'
```

The result is shown in the plot below:



Note that y_1 and y_2 are quite close. The major difference is that the step response of the closed-loop system is flatter near $t = 0$, which is a result of the fourth order dynamics of the system

Problem 3: A unity gain negative feedback system has an open-loop transfer function given by

$$G(s) = \frac{K(1+5s)}{s(1+10s)(1+s)^2}$$

Draw a Bode diagram for this system and determine the loop gain K required for a phase margin of 20 deg. What is the gain margin?

A lag compensator

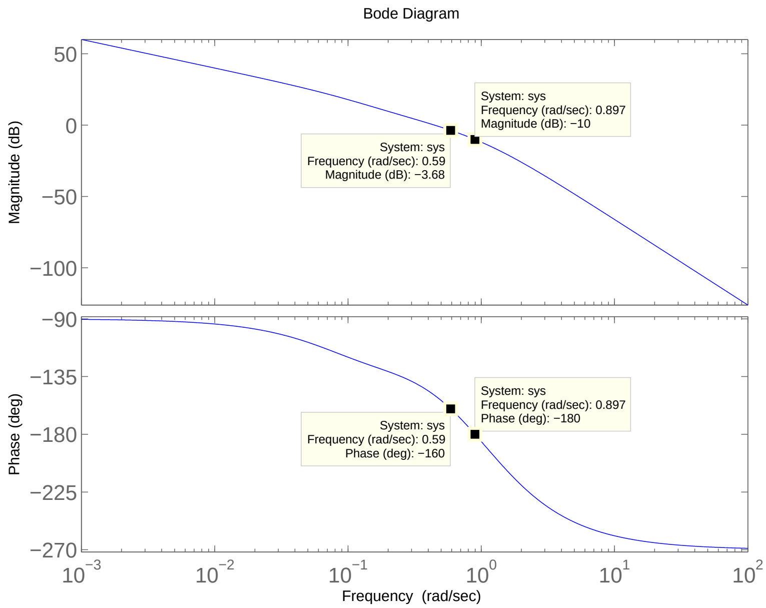
$$G_c(s) = \frac{1+10s}{1+50s}$$

is added to this system. Use Bode diagrams to find the reduction in steady state error following a ramp change to the reference input, assuming that the 20 deg phase margin is maintained.

Solution: Using Matlab, plot the Bode plot:

```
>> num = [0 0 0 5 1];
>> den = conv([10 1 0],[1 2 1])
den =
10 21 12 1 0
>> bode(num,den)
>> w = logspace(-3,2,3000);
>> bode(num,den,w)
```

The Bode plot is shown below:



Using the cursor tool, I found that the phase is -160 deg (20 deg phase margin) at $\omega = 0.59$. Find the magnitude at that frequency:

```
>> [mag,phase] = bode(num,den,0.59)
mag =
0.6544
phase =
-160.1873
```

So to get crossover at that frequency, must chose

$$K = \frac{1}{0.6544} = 1.528$$

Next, use the cursor tool to find the frequency at which the phase is -180 deg, to find the gain margin. The phase is -180 deg at $\omega = 0.897$, at which frequency the gain of the original system is 0.3145:

```
>> [mag,phase] = bode(num,den,0.897)
mag =
0.3145
phase =
-179.9924
```

So the gain of the system with the new gain K included at $\omega = 0.897\,\mathrm{r/s}$ is $1.528 \times 0.3145 = 0.481$. The gain margin is the reciprocal of this number, so that

$$\text{Gain margin} = 2.08$$

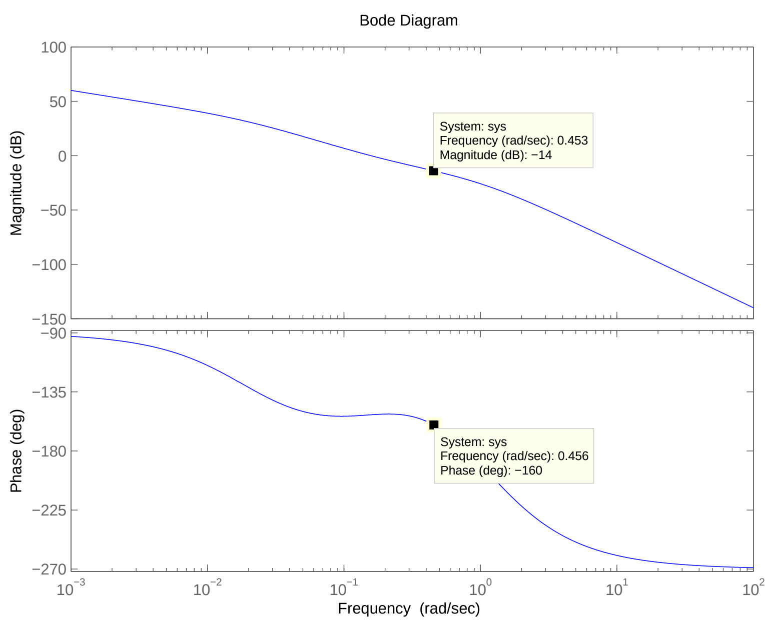
Now add the lag compensator

$$G_c(s) = \frac{1 + 10s}{1 + 50s}$$

The new loop transfer function numerator and denominator (again assuming $K = 1$) is found in Matlab via

```
>> num2 = conv(num,[10 1])
num2 =
0 0 0 50 15 1
>> den2 = conv(den,[50 1])
den2 =
500 1060 621 62 1 0
```

The bode plot is below:



Plotting the Bode plot and using the cursor tool yields a crossover frequency of $\omega_c = 453$ r/s to achieve a phase margin of 20 deg. At this frequency, the magnitude of the transfer function with gain $K = 1$ is 0.2, so we need $K = 5$ to get the desired crossover frequency.

For the original control system with $K = 1.528$ and no lag compensator, the velocity constant is

$$K_v = \lim_{s \to 0} sG(s) = K = 1.528$$

For the modified system with $K = 5$ and the lag compensator, the velocity constant is

$$K_v = \lim_{s \to 0} sG(s)G_c(s) = K = 5$$

Therefore, the steady-state error to a unit ramp is reduced from $1/K_v = 0.654$ for the original control system, to $1/K_v = 0.2$ for the modified control system, a reduction a factor of 3.27.

Problem 4: Plot the Nyquist diagram for the plant with the unstable open-loop transfer function

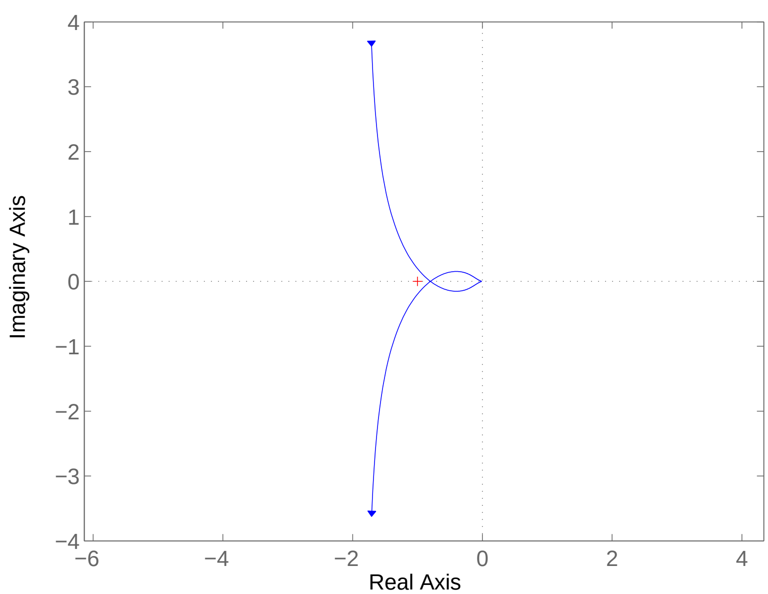
$$G(s) = \frac{K(s+0.4)}{s(s^2+2s-1)}$$

Determine the range of K for which the closed-loop system with unity negative gain feedback which incorporated this plant would be stable.

Solution: This system has one unstable pole (at $s = 0.4142$), an we need therefore one counter clockwise encirclement of the point $-1/K$ for stability. Use Matlab to plot the Nyquist diagram:

```
>> num = [1 0.4];
>> den = [1 2 -1 0];
>> nyquist(num,den,.1:.01:10);axis equal
>> title('')
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = xlabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> h = gca;
>> set(h,'fontsize',14);
>> h = xlabel('Real Axis');
>> set(h,'fontsize',14);
>> h = ylabel('Imaginary Axis');
>> set(h,'fontsize',14);
>> print -depsc 'figure6.eps'
```

The nyquist plot is below:



The closure of the Nyquist plot is a bit tricky. Since the Nyquist contour in the s-plane must detour around the pole at $s = 0$ in a counterclockwise direction, the Nyquist plot above closes in a clockwise direction. So to get one CCW encirclement, $-1/K$ must be inside the small loop near the origin. Blowing up the plot shows that the real line inside the loop is

$$-0.8 < -1/K < 0$$

which implies that

$$K > \frac{1}{0.8}1 = 1.25$$

Indeed, the characteristic equation for $K = 1.25$ is

$$\phi(s) = (s^3 + 2s^2 - s) + 1.25\,(s + 0.4) = s^3 + 2s^2 + 0.25s + 0.5 = 0$$

which has roots at

$$s = -2, \quad s = \pm 0.5j$$

That is, for $K = 1.25$, the closed-loop system is at the boundary of stability, with oscillatory poles on the $j\omega$-axis, as is predicted by Nyquist.