16.31 Homework 2 — Solution

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Issued: September 22, 2006
Due: September 29, 2006

Problem 1. (Dominant Pole Locations) [FPE 3.36 (a),(c),(d), page 161]. Consider the second order system

$$H(s) = \frac{\omega_n^2}{(s/p + 1)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

The unit step response is

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\sin(\omega_d t - \theta)$$

where

$$A = \frac{-\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2} \qquad B = \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)(1-\zeta^2)}}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{p - \zeta\omega_n}$$

(a) State the steps that you would follow to show that this is the step response. Which inverse transforms would you use from the Tables?

(b) Which term dominates $y(t)$ as p gets large?

(c) Which term dominates as p gets small? (small with respect to what?)

(d) Using the explicit expression for $y(t)$ above and the *step* command in Matlab (assume $\omega_n = 1$ and $\zeta = 0.7$), plot the step response for various values of p ranging from very small to very large. At what point does the extra pole cease to have an impact on the system response?

Solution. (a) To find the inverse LT, do a partialfraction expansion, into either exclusively first-order terms (in which case some of the residues will be complex), or into first- and second-order terms. In many ways, it is easier to do all first-order terms. First, express $H(s)$ in factored form as

$$H(s) = \frac{p\omega_n^2}{(s+p)\left(s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)\right)\left(s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)\right)}$$

The step response has LT given by $H(s)/s$, since the LT of a step is $1/s$. Therefore,

$$Y(s) = \frac{p\omega_n^2}{s(s+p)\left(s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)\right)\left(s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)\right)}$$

Then expand in a "partial fraction expansion,"

$$H(s) = \frac{c_1}{s} + \frac{c_2}{s+p} + \frac{c_3}{s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)} + \text{c.c.}$$

where "c.c." denotes the complex conjugate of the previous term. Find the constants using the coverup method, so that

$$c_1 = \lim_{s\to 0} sY(s) = 1$$

$$c_2 = \lim_{s\to -p} (s+p)Y(s) = \frac{-\omega_n^2}{p^2 - 2\zeta\omega_n p + \omega_n^2}$$

$$\begin{aligned} c_3 &= \lim_{s\to -\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n} \left(s - \left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)\right) Y(s) \\ &= \frac{-p}{2\sqrt{1-\zeta^2}\left(j\zeta + \sqrt{1-\zeta^2}\right)\left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n + p\right)} \end{aligned}$$

Then the total response is given by

$$y(t) = c_1 + c_2 e^{-pt} + 2\Re\left[c_3 \exp\left(\left(-\zeta\omega_n + j\sqrt{1-\zeta^2}\omega_n\right)t\right)\right]$$

where $\Re[\cdot]$ denotes the real part of $[\cdot]$. This last expression can be simplifed considerably, as

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\cos(\omega_d t - \phi)$$

where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The magnitude B is given by

$$B = 2\,|c_3| = \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)\,(1-\zeta^2)}}$$

and the phase ϕ is given bY

$$\phi = \arg(c_3) = \pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) - \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{p - \zeta\omega_n}\right)$$

Trig identities may be used to give the form in the problem statement. (Note the small typo in the problem statement.)

(b) As p gets large, c_2 and A go to zero, so that the first-order term is neglible, and the oscillatory term dominates.

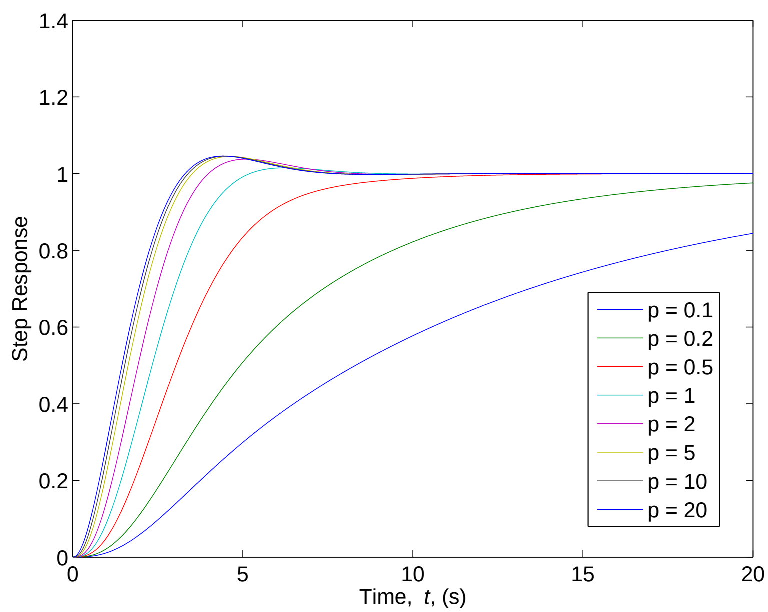
(c) As p gets small (with respect to ω_n), c_b and B go to zero, so that the oscillatory term is neglible, and the first-order term dominates.

(d) I used the Matlab commands

```
wn = 1;
zeta = 0.7;
num = wn^2;
```

```
P = [0.1 0.2 0.5 1 2 5 10 50];
Y=[];
for p = P;
  den = conv([1/p 1],[1 2*zeta*wn^2 wn^2]);
  t = 0:.01:20 ;
  y = step(num,den,t);
  Y = [Y y];
end
plot(t,Y)
legend('p = 0.1','p = 0.2','p = 0.5','p = 1','p = 2','p = 5','p = 10','p = 20')
figure(1)
title('')
h = gca;
set(h,'fontsize',14)
h = xlabel('Time, {\it t}, (s)');
set(h,'fontsize',14)
h = ylabel('Step Response')
set(h,'fontsize',14)
```

The result is the plot below:



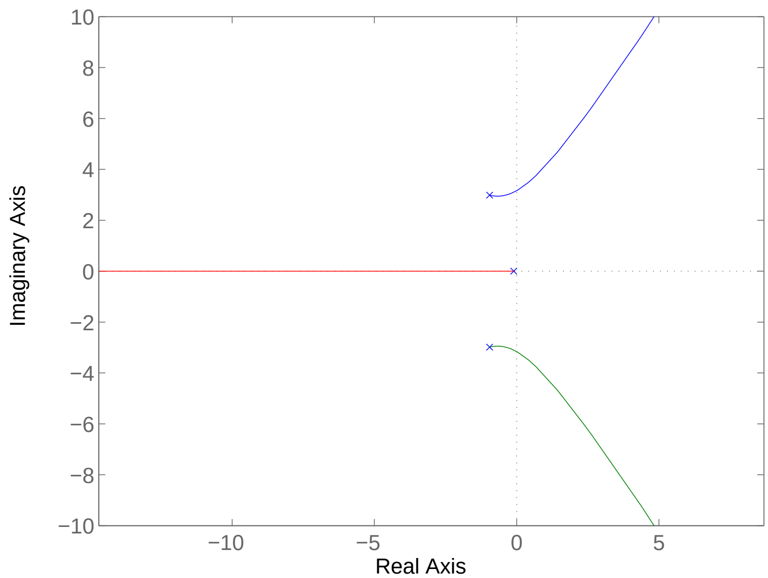
Note that as p gets larger than about 2, changing p has little effect.

Problem 2. (Basic Root Locus Plotting) [FPE problem 5.4 (a), (e), and problem 5.7 (e)] Sketch the root locus for the following systems. As we did in class, concentrate on the real-axis, and the asymptotes/centroids.

Solution. For each of these, I just did the Matlab plot.

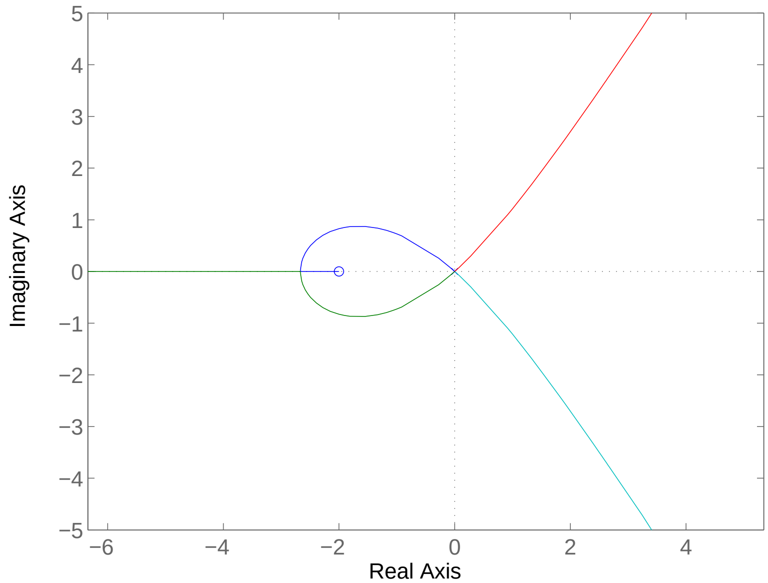
(a) $G_cG(s) = \dfrac{K}{s(s^2+2s+10)}$. Use the Matlab script

```
num = 1;
den = [1 2 10 1];
rlocus(num,den)
title('')
h = gca;
set(h,'fontsize',14)
h = xlabel('Real Axis');
set(h,'fontsize',14)
h = ylabel('Imaginary Axis');
set(h,'fontsize',14)
axis equal
print -depsc 'figure2a.eps'
```



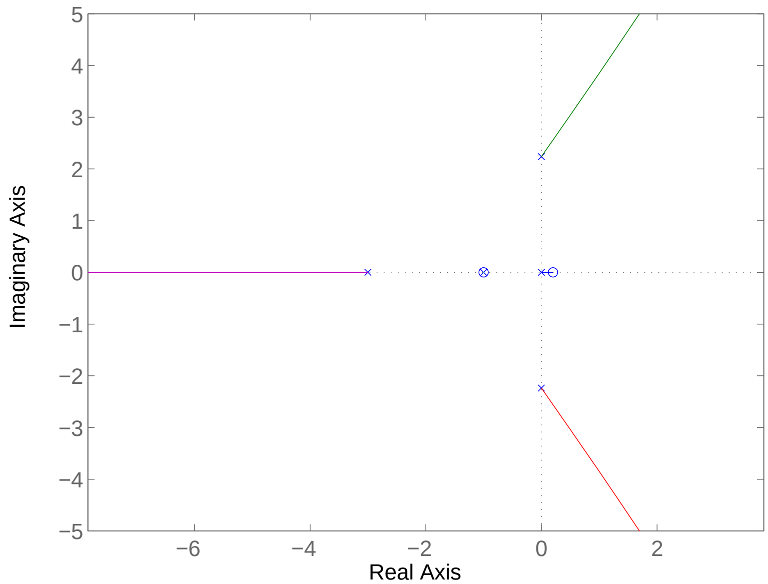
(b) $G_cG(s) = \dfrac{K(s+2)}{s^4}$. Matlab code:

```
num = [1 2];
den = [1 0 0 0 0];
...
```



(c) $G_cG(s) = \dfrac{K(s+1)(s-0.2)}{s(s+1)(s+3)(s^2+5)}$. Matlab:

```
num = conv([1 1],[1 -0.2]);
den = conv([1 1 0],conv([1 3],[1 0 5]));
...
```



Problem 3. The attitude-control system of a space booster is shown in Figure 2. The attitude angle θ is controlled by commanding the engine angle δ, which is then the angle of the applied thrust, F_T. The vehicle velocity is denoted by v. These control systems are sometimes open-loop unstable, which occurs if the center of aerodynamic pressure is forward

of the booster center of gravity. For example, the rigid-body transfer function of the Saturn V booster was

$$G_p(s) = \frac{0.9407}{s^2 - 0.0297}$$

This transfer function does not include vehicle bending dynamics, liquid fuel slosh dynamics, and the dynamics of the hydraulic motor that positioned the engine. These dynamics added 25 orders to the transfer function! The rigid-body vehicle was stabilized by the addition of rate feedback, as shown in the Figure 2b (Rate feedback, in addition to other types of compensation, was used on the actual vehicle.)

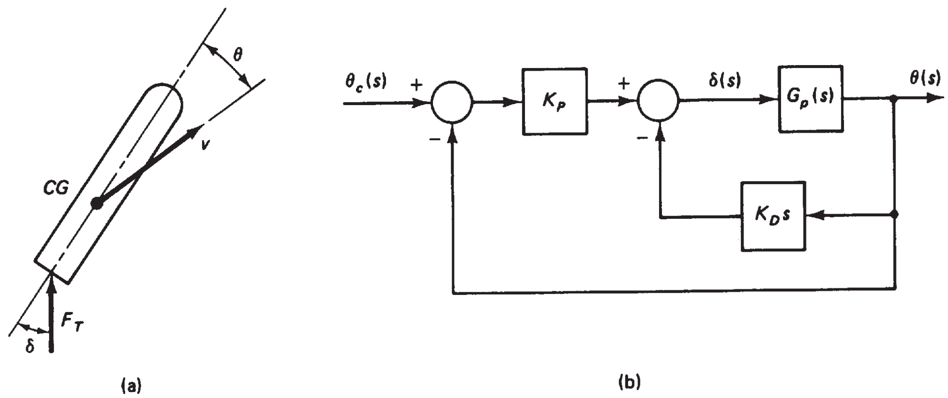


Figure 1: Booster control system

(a) With $K_D = 0$ (the rate feedback removed), plot the root locus and state the different types of (unstable) responses possible (relate the response with the possible pole locations)

(b) Design the compensator shown (which is PD) to place a closed-loop pole at $s = -0.25 + j0.25$. Note that the time constant of the pole is 4 sec, which is not unreasonable for a large space booster.

(c) Plot the root locus of the compensated system, with K_p variable and K_D set to the value found in (b).

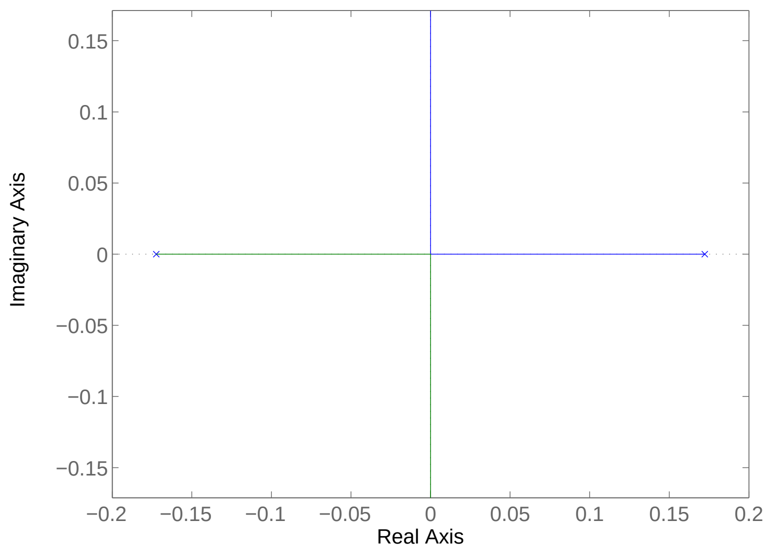
(d) Use Matlab to compute the closed-loop response to an impulse for θ_c.

Solution. (a) Use Matlab to plot this (simple) locus:

```
num = [0 0 0.9407];
den = [1 0 -0.0297];
rlocus(num,den)
title(”)
h = gca;
set(h,’fontsize’,14)
h = xlabel(’Real Axis’);
```

```
set(h,'fontsize',14)
h = ylabel('Imaginary Axis');
set(h,'fontsize',14)
axis equal
print -depsc 'figure3a.eps'
```

The result is kind of boring:



For low gains, the poles are at $s = \pm a$, for some posistive a, so there is a real, unstable pole, corresponding to an exponentially growing output. This corresponds to an exponentiallly increasing angle θ, as the booster "loses it" and begins to tumble. For high gains, the poles are of the form $s = \pm ja$, for positive a, so that the angle θ oscillates over time. So position feedback using a static gain is insufficent for this problem.

(b) Using a PD controller, the loop gain is

$$(K_D s + K_P)\,\frac{0.9407}{s^2 - 0.0297}$$

so the characteristic polynomial is

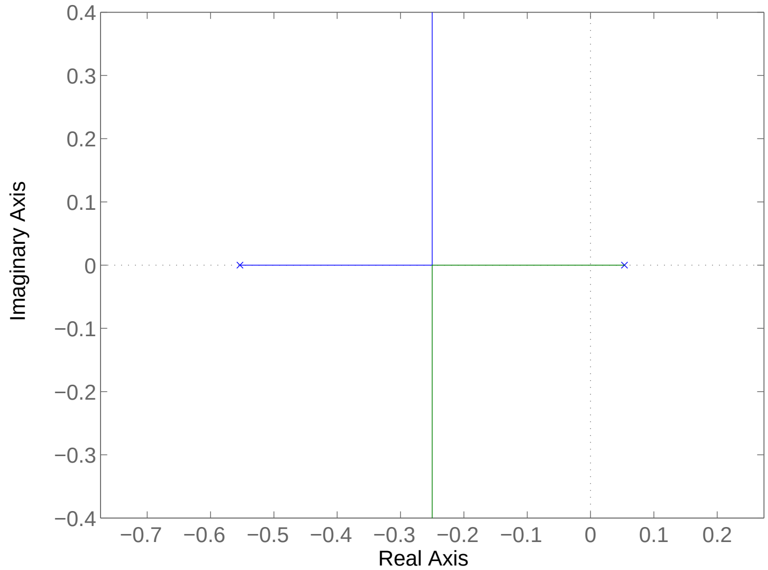
$$\phi(s) = s^2 - 0.0297 + 0.9407(K_D s + K_P)$$

We want $\phi(s) = s^2 + 0.5s + 0.125$. Hence, we must chose $K_P = 0.1645$, $K_D = 0.5315$.

(c) With $K_D = 0.5315$, K_P variable, the characteristic equation is

$$\phi(s) = s^2 + 0.5s - 0.0297 + 0.9407\,K_P$$

The root locus corresponds to a system with no zeros, and poles at $s = -0.5536$ and $s = 0.0536$. The root locus is as shown below:



The root locus is not that different from the one in part (a), just shifted to the left by 0.25.

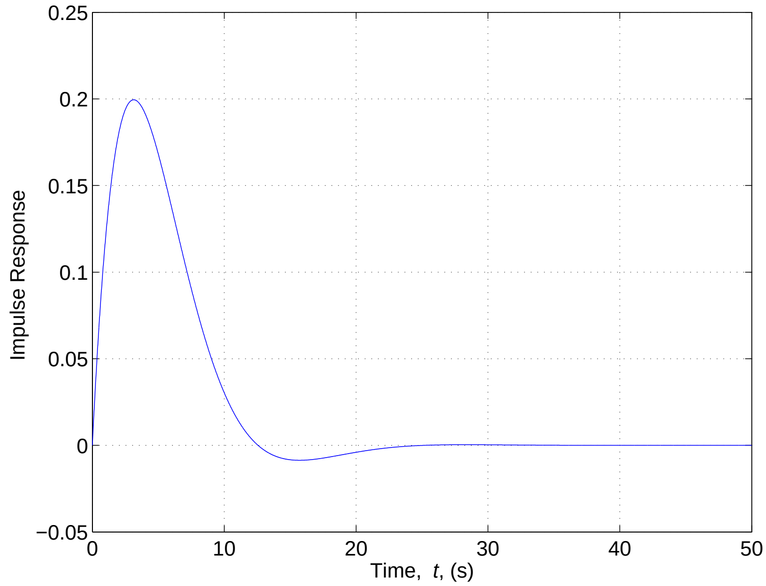
(d) For the loop as shown, the transfer function is

$$H(s) = \frac{K_P G_p(s)}{1 + (K_D s + K_P)\, G_p(s)} = \frac{0.1547}{s^2 + 0.5s + 0.125}$$

Using the Matlab code

```
>> num = [0 0 0.1547];
>> den = [1 0.5 0.125];
>> t=0:0.01:50;
>> y=impulse(num,den,t);
>> plot(t,y)
>> axis([0 50 -0.05 0.25])
>> grid
>> title('')
>> h = gca;
>> set(h,'fontsize',14)
>> h = xlabel('Time, {\it t}, (s)');
>> set(h,'fontsize',14)
>> h = ylabel('Impulse Response ');
>> set(h,'fontsize',14)
>> print -depsc 'figure3d.eps'
```

the impulse response is found to be:


0.25
0.2
0.15
0.1
0.05
0
−0.05
Impulse Response
0
10
20
30
40
50
Time, t, (s)