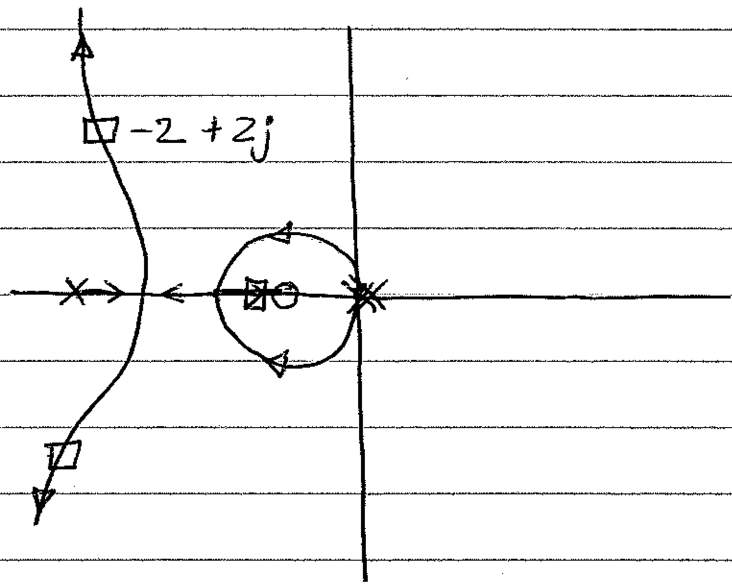
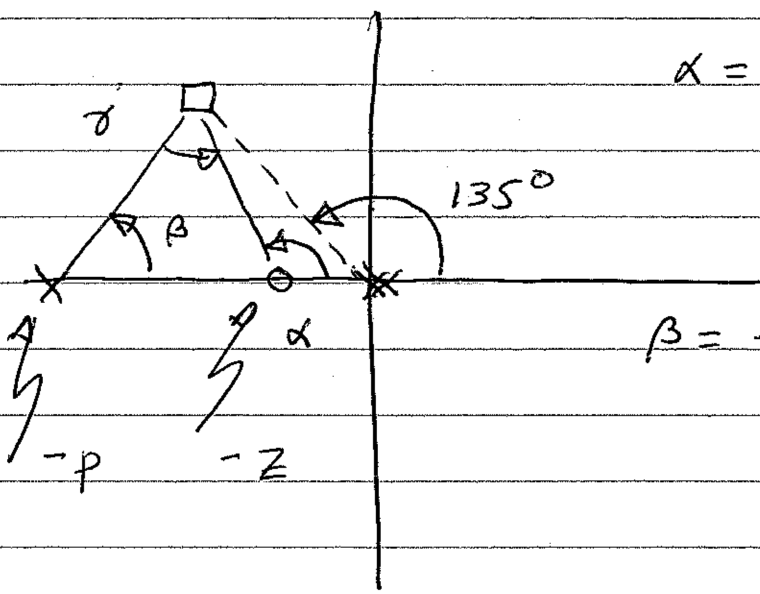
HW3 Solution

1. Root locus will be



Look at angle condition:



$$\alpha = \tan^{-1} \frac{2}{-2-z}$$

$$\beta = \tan^{-1} \frac{2}{-2-p}$$

Need to have

$$-2 \times 135^\circ - \beta + \alpha = -180^\circ$$

There are lots of possible choices.
One approach is to choose α, β to get min. lead ratio p/z.
A little iteration gives

$$z = -1.2, \quad p = -7$$

So the compensator is

$$G_c(s) = k \frac{s+1.2}{s+7}$$

The magnitude condition is that

$$k = \left. \frac{1}{|G_p(s)| \left| \frac{s+1.2}{s+7} \right|} \right|_{s=-2+2j}$$

$$= 20$$

So the compensator is

$$G_c(s) = \frac{20(s+1.2)}{s+7}$$

The CL poles are then at $s = -2 \pm 2j, -3$

2. To get P.O. $\leq 5\%$, use

$$P.O. = \exp(-\pi / \tan\theta) = 0.05$$

$$\Rightarrow \theta = 0.8092 \text{ rad}$$

$$\zeta = \cos\theta = 0.69$$

To get $\zeta \approx 0.69$, require $\boxed{P.M. \approx 69^\circ}$

Also, want $t_r \leq 0.15$ sec. Use

$$t_r \approx 1.8/\omega_c$$

to obtain $\boxed{\omega_c = 12 \text{ rad/s}}$.

Now, at $\omega = 12$ rad/s, $G(j\omega)$ has phase -128°, and thus only 52° phase margin if we just use gain feedback. So instead, use lead compensator, with 20° lead to bring P.M. to 72°. To get 20° lead, need a lead ratio of about 2. So take

$$G_c(s) = k \frac{s + 8.5}{s + 17}$$

Assuming $K=1$, need $k = 8777$ to get $\omega_c = 12$ rad/sec.

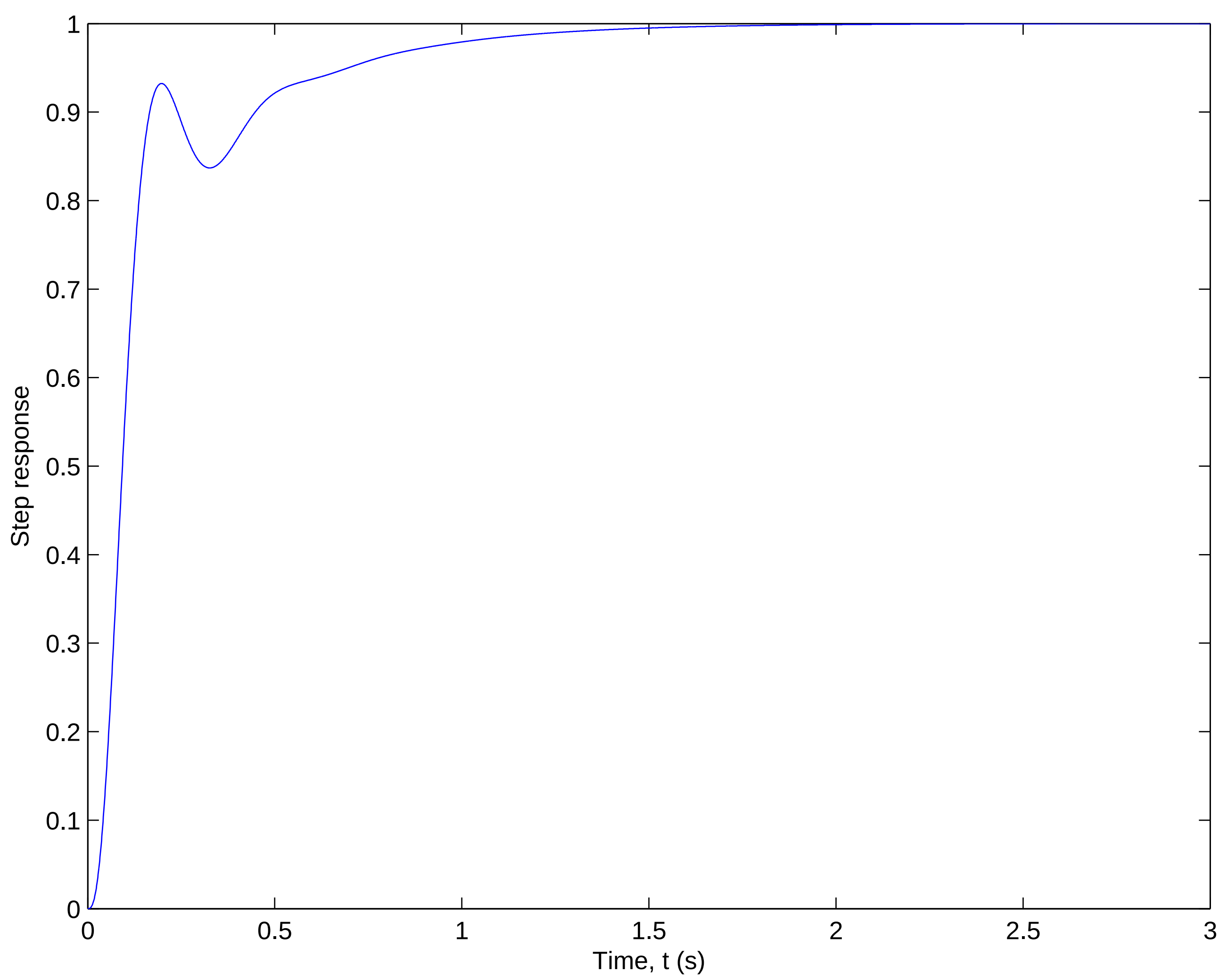
Unfortunately, the resulting C.L system has P.O. of about 27%, due to the very low $K_p \approx 1.96$. (The approximation $\zeta \approx PM/100°$ only works for reasonable K_p.) It is very difficult to achieve the desired specs, without going to a much higher ω_c, and using lots of lead.

The problem is that the plant has 3 poles, so lots of phase w/o much magnitude rolloff. To get a good loop shape, need plant to be more like $1/s^2$, but this means cancelling poles with zeros (PD control), which may cause problems. Short of cancelling the plant dynamics, the best I could do was

$$G_c(s) = 17{,}000 \cdot \frac{(s+4)(s+8)}{s(s+40)}$$

This gives $\omega_c = 12$ r/s, $PM = 73°$, a type I system (no steady error), but a funny response, with a long tail, which is typical of a system with lag compensation. However, there is no overshoot, and $t_r = 0.12s$.

See response next page.



Step response
Time, t (s)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3

3. $$G(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)}$$

$$= \frac{s^2 + 3s + 2}{s^2 + 7s + 12}$$

$$= \frac{-4s - 10}{s^2 + 7s + 12} + 1$$

The simplest choice is controller canonical form, so that

$$A = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & -10 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \end{bmatrix}$$

Alternatively,

$$G(s) = 1 - \frac{+4s + 10}{(s+3)(s+4)}$$

$$= 1 + \frac{2}{s+3} + \frac{-6}{s+4}$$

A modal s.s. description is

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 \end{bmatrix}$$

That, in each case, $G(s) = C(sI - A)^{-1}B + D$ can be confirmed by Matlab or direct substitution.

4. $$\dot{x}_1 = x_1(u - \beta x_2)$$

$$\dot{x}_2 = x_2(-\alpha + \beta x_1)$$

(a) This system is <u>nonlinear</u>, since $\dot{x}_1$ has $x_1 x_2$ term. It is <u>time-invariant</u>, since there is no t dependency on RHS.

(b) At equilibrium, $\dot{x}_1 = \dot{x}_2 = 0$

$$\Rightarrow \quad 0 = x_1(u - \beta x_2) = x_1 - \beta x_1 x_2$$
$$0 = x_2(-\alpha + \beta x_1) = -\alpha x_2 + \beta x_1 x_2$$

So the equilibrium is either

$$x_1 = 0, \quad x_2 = 0, \quad u = 1$$

$$x_2 = \frac{1}{\beta}, \quad x_1 = \frac{\alpha}{\beta}, \quad u = 1$$

(c) Use the 2nd equilibrium point above. Then

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{eq.} \qquad B_i = \frac{\partial f_i}{\partial u}$$

$$A_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{eq} = \left. u - \beta x_2 \right|_{eq} = 0$$

$$A_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{eq} = \left. -\beta x_1 \right|_{eq} = -\alpha$$

$$A_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{eq} = \left. \beta x_2 \right|_{eq} = 1$$

$$A_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{eq} = \left. -\alpha + \beta x_1 \right|_{eq} = 0$$

$$B_1 = \left.\frac{\partial f_1}{\partial u}\right|_{eq} = \left. x_1 \right|_{eq} = \frac{1}{\beta}$$

$$B_2 = \left.\frac{\partial f_2}{\partial u}\right|_{eq} = 0$$

Therefore,

$$\delta \dot{\underline{x}} = \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix} \delta \underline{x} + \begin{bmatrix} 1/\beta \\ 0 \end{bmatrix} \delta u$$