16.31 Homework 4

Prof. S. R. Hall   Issued: October 16, 2006
                                                            Due: October 23, 2006

1. (FPE 6.47) State the Cayley-Hamilton theorem. Verify that it is true on a random 5×5 matrix in Matlab using `polvalm`.

2. The goal is to prove that the condition for controllability is that

$$\text{rank}\, \mathcal{M}_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$$

   To proceed, start with the orthogonality condition:
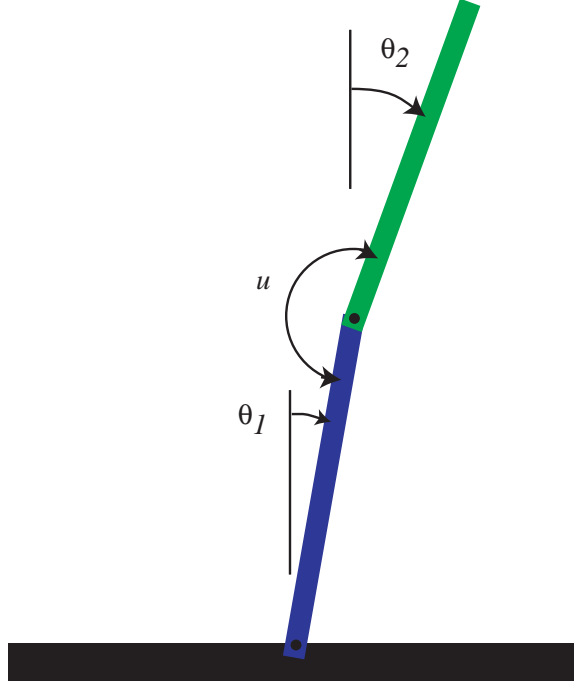
$$(x^\star)^T e^{At} B = 0$$

   and expand the matrix exponential as a power series. Then show that the orthogonality condition can be rewritten as the product of three matrices

$$(x^\star)^T M_1 M_2(t) = 0$$

   where the entire time dependence of the matrix exponential is embedded into $M_2(t)$. The basic test is derived by noting that this orthogonality condition must hold for all time t, and confirm that (since $x^\star \neq 0$) this condition can then only be true if M_1 is rank deficient.

   At this point, M_1 and M_2 will have an infinite number of columns and rows, respectively. Show how to use the Cayley-Hamilton theorem to convert the infinite matrices to finite ones, thereby recovering the controllability condition given above.

3. The *acrobot* is a robotic arm consisting of two links, as shown in the figure below:

The first link (in blue) is hinged at the "shoulder," and connected to a second link (in green) at the "elbow." Unlike a human arm, the acrobot is actuated only at the elbow, which applies equal and opposite toques, with magnitude u, to the two links. The arm is, in the parlance of robotics, underactuated, since only one of the two joints is directly actuated. Each link has mass m_1 or m_2, and length L_1 or L_2. If each link is uniform, then the CG of each link is at its midpoint, and the moment of inertia is $I_1 = m_1 L_1^2/12$, and similarly for the second link. The coordinates describing the configuration of the acrobot are the angles θ_1 and θ_2 of each link from the vertical. If we define the states

$$x_1 = \theta_1, \quad x_2 = \dot{\theta}_1, \quad x_3 = \theta_2, \quad x_4 = \dot{\theta}_2$$

and simplify by setting all the parameters of the problem to unity (m_1, m_2, L_1, L_2, and g), then the state dynamics become $\dot{x}_i = f_i(x_1, x_2, x_3, x_4, u)$, where

$f_1 = x_2$

$f_2 = \dfrac{-24u - 36u\cos(x_1 - x_3) + 27\sin(x_1) + 9\sin(x_1 - 2x_3) - 9x_2^2\sin(2x_1 - 2x_3) - 12x_4^2\sin(x_1 - x_3)}{23 - 9\cos(2x_1 - 2x_3)}$

$f_3 = x_4$

$f_4 = \dfrac{96u + 36u\cos(x_1 - x_3) - 27\sin(2x_1 - x_3) + 21\sin x_3 + 48x_2^2\sin(x_1 - x_3) + 9x_4^2\sin(2x_1 - 2x_3)}{23 - 9\cos(2x_1 - 2x_3)}$

We will consider two equilibrium configurations. In configuration 1, the nominal configuration is $\theta_1 = \theta_2 = 0$, and the nominal input is $u_0 = 0$. That is, the acrobot is

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vertical. In configuration 2, the nominal configuration is $\theta_1 = 0.339837\ldots$, $\theta_2 = -\pi/2$, and $u_0 = -1/2$. That is, the acrobot is in the shape of a figure 7.

For these two configurations, the state and control matrices are

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 18/7 & 0 & -9/7 & 0 \\ 0 & 0 & 0 & 1 \\ -27/7 & 0 & 24/7 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -30/7 \\ 0 \\ 66/7 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.13137\ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.565685\ldots & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.35697\ldots \\ 0 \\ 3.45720\ldots \end{bmatrix}$$

For each configuration,

(a) Find the controllability matrix. Show that in each case the system is controllable.

(b) Find the eigenvalues and eigenvectors of the system.

(c) Find a the state feedback gain matrix K that places all of the closed-loop poles of each system at $s = -1.5$.

(d) For each configuration, find and plot the homogeneous response of the system, under closed-loop control, to the initial condition

$$\delta x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Is the response acceptable? Explain. What are the implications for controlling this system?

(e) I would claim that configuration 1 is not very controllable. In class, we gave a yes/no definition of controllability, so this statement is a little soft. Nevertheless, using the controllability matrix for configuration 1, argue that configuration1 is not very controllable. Hint: Can you argue that the controllability matrix is nearly singular? How?

4. We have talked about selecting the feedback gains to change the pole locations of the system, but we have not mentioned anything about how the open-loop zeros are changed. Part of the reason for this is that it can be shown that:

When full state feedback is used $(u = \bar{N}r - Kx)$ to control a system, the zeros remain unchanged by the feedback.

Confirm that this statement is true by analyzing the zero locations for the closed-loop system, which are given by the roots of the polynomial:

$$\det \begin{bmatrix} sI - (A - BK) & -\bar{N}B \\ C & 0 \end{bmatrix} = 0$$

The best way to proceed is to show that, through a series of column and row operations that **do not** change the value of the determinant, you can get the following reduction:

$$\det \begin{bmatrix} sI - (A - BK) & -B\bar{N} \\ C & 0 \end{bmatrix} = 0 \quad \Rightarrow \quad \det \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} = 0$$

which, of course, is the same polynomial used to find the open-loop zeros of the system.