16.31 HW4 SOLUTION.

1. The Cayley-Hamilton theorem states that every matrix satisfies its own characteristic equation. That is, if

$$\det(sI - A) = \phi(s)$$

then $\phi(A) = 0$. An example of this on a random matrix A is given on the next page.

```
>> A=randn(5,5)
A =
   -0.4326    1.1909   -0.1867    0.1139    0.2944
   -1.6656    1.1892    0.7258    1.0668   -1.3362
    0.1253   -0.0376   -0.5883    0.0593    0.7143
    0.2877    0.3273    2.1832   -0.0956    1.6236
   -1.1465    0.1746   -0.1364   -0.8323   -0.6918
>> d=eig(A)
d =
  -0.5521 + 1.7693i
  -0.5521 - 1.7693i
   0.8022 + 0.8770i
   0.8022 - 0.8770i
  -1.1193
>> p=poly(d)
p =
    1.0000    0.6191    2.5168   -0.5083    0.4302    5.4319
>> polyvalm(p,A)
ans =
   1.0e-13 *
   -0.0533    0.1262   -0.1769   -0.0296   -0.2160
    0.0413    0.0888    0.1038    0.2348   -0.1741
   -0.0739    0.0432    0.0178   -0.0293    0.0201
   -0.1534    0.1692    0.1310   -0.0266    0.1341
   -0.1375   -0.0314    0.0181    0.0132   -0.1332
```

2. A system is uncontrollable if there exists a direction x^* such that

$$(x^*)^T e^{At} B = 0$$

for all time.

The matrix exponential is given by

$$e^{At} = \sum_{n=1}^{\infty} A^n t^n / n!$$

Therefore,

$$e^{At} B = \sum_{n=1}^{\infty} A^n B t^n / n!$$

This product can be written as

$$e^{At} B = [B \quad AB \quad A^2B \quad \cdots] \begin{bmatrix} I \\ tI \\ t^2 I/2 \\ \vdots \end{bmatrix}$$

$$= M_1 M_2(t)$$

where I has as many rows and columns as B has columns.

Now, the Nth column is $A^N B$. By the Cayley-~~Hamit~~ Hamilton theorem, A^N can be expressed as a linear combination of $I, A, A^2, \ldots A^{N-1}$, where N is the dimension of A. More specifically, if

$$\det(sI - A) = \phi(s) = s^N + a_{N-1}s^{N-1} + \cdots + a_0,$$

then $\phi(A) = 0$, so that

$$A^N = -a_{N-1}A^{N-1} - a_{N-2}A^{N-2} - \cdots - a_0 I$$

Likewise, A^{N+1} is a linear combination of the $A^N, A^{N-1}, \ldots A$, and therefore is also a linear combination of $A^{N-1}, A^{N-2}, \ldots I$. As a result, M_1 may be expressed as

$$M_1 = \underbrace{\begin{bmatrix} B & AB & \cdots & A^{N-1}B \end{bmatrix}}_{M_c} \underbrace{\begin{bmatrix} I & 0 & & 0 & -a_0 I & \\ 0 & I & & 0 & -a_1 I & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ 0 & 0 & & I & -a_{N-1} I & \end{bmatrix}}_{M_3}$$

Therefore,

$$M_1 M_2(t) = [M_c M_3] M_2(t) = M_c [M_3 M_2(t)]$$

$$\Rightarrow M_1 M_2(t) = M_c M_4(t)$$

where $M_4(t) = M_3 M_2(t)$, and thus

$$(x^*)^T e^{At} B = (x^*)^T M_c M_4(t)$$

If M_c has rank less than N, then there exists an x^* such that $(x^*)^T M_c = 0$, and therefore $(x^*)^T e^{At} B = 0$ for all time. Conversely, if rank $M_c = N$, then $(x^*)^T M_c \neq 0$ (for $x^* \neq 0$), and since the rows of $M_4(t)$ are linearly independent, $(x^*)^T e^{At} B \neq 0$ for every t.

The net result is that (A, B) controllable iff rank $M_c = N$.

3. In the following pages, see the Matlab output and plots for parts (a)–(d). In addition,

(d) In both cases, small initial perturbations in the IC (~ 1 deg) can cause large perturbations (≈ 15 deg) in the response. So the C.L. response is not likely to be acceptable – it is very likely that small perturbations will cause the state to go outside the linear range.

(e) The 1st column of M_{cl} is

$$(0, -30/7, 0, 66/7)^T$$

The 3rd column is

$$(0, -162/7, 0, 342/7)^T$$

These vectors are nearly colinear, since

$$\frac{66/7}{-30/7} \approx -2.2, \quad \frac{342/7}{-162/7} \approx -2.111$$

Hence, M_{c1} is nearly singular.

```
echo on
%
% The A and B matrix for configuration 1
%
A1 = [0 1 0 0;18/7 0 -9/7 0;0 0 0 1;-27/7 0 24/7 0]

A1 =

         0    1.0000         0         0
    2.5714         0   -1.2857         0
         0         0         0    1.0000
   -3.8571         0    3.4286         0

B1 = [0;-30/7;0;66/7]

B1 =

         0
   -4.2857
         0
    9.4286

%
% (a) The controllability matrix is
%
Mc1 = [B1 A1*B1 A1*A1*B1 A1*A1*A1*B1]

Mc1 =

         0   -4.2857         0  -23.1429
   -4.2857         0  -23.1429         0
         0    9.4286         0   48.8571
    9.4286         0   48.8571         0

%
% The controllability matrix is full rank:
%
rank(Mc1)

ans =

     4

%
% (b) The eigenvalues and vectors are:
%
```

```
[v,d]=eig(A1);d=diag(d);
%
% The eigenvalues:
%
d

d =

    2.2952
   -2.2952
    0.8557
   -0.8557

%
% The eigenvectors:
%
v

v =

   -0.1719   -0.1719   -0.4353   -0.4353
   -0.3946    0.3946   -0.3725    0.3725
    0.3605    0.3605   -0.6227   -0.6227
    0.8275   -0.8275   -0.5329    0.5329

%
% (c) Find the gain K to place all the poles at s=-1.5:
%
K1 = acker(A1,B1,[-1.5,-1.5,-1.5,-1.5])

K1 =

  -18.3516  -19.6875   -6.2734   -8.3125

%
% Check the results:
%
eig(A1-B1*K1)

ans =

  -1.5011
  -1.5000 + 0.0011i
  -1.5000 - 0.0011i
  -1.4989
```

```
%
% (d) Find the homogenous response. There are a number of ways
to do this
% in Matlab. The easiest is that the homogenous response to an
initial
% condition x0 is the same as the impulse response for the
system (A1,x0,C).I
% will find and plot just the position variables, x1 and x3:
%
C = [1 0 0 0;0 0 1 0]

C =

     1     0     0     0
     0     0     1     0

D = [0;0]

D =

     0
     0

x0 = [1;0;0;0]

x0 =

     1
     0
     0
     0


sys = ss(A1-B1*K1,x0,C,D)

a =
            x1       x2       x3       x4
   x1        0        1        0        0
   x2   -76.08   -84.37   -28.17   -35.62
   x3        0        0        0        1
   x4    169.2    185.6    62.58    78.37


b =
        u1
   x1    1
```

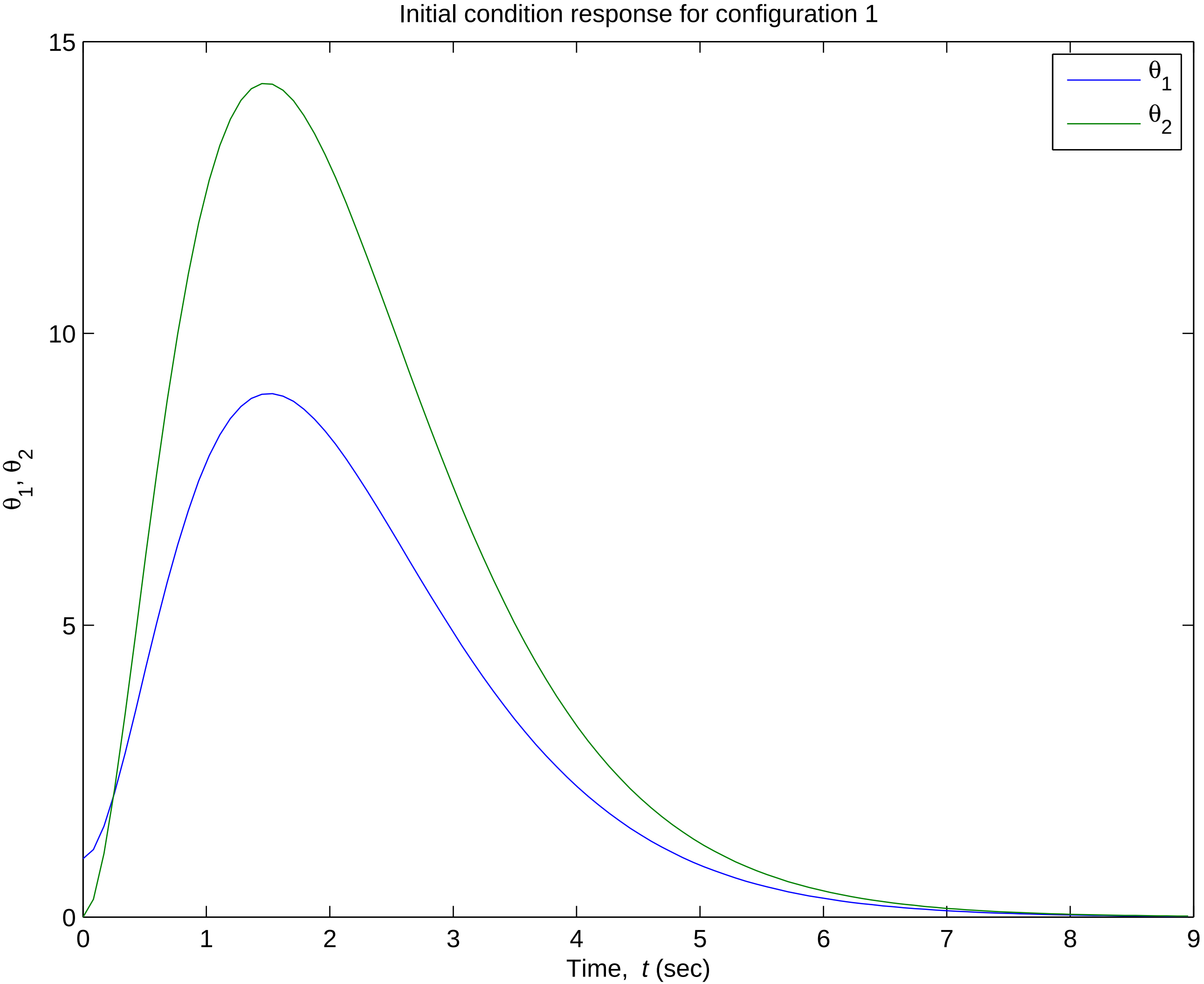
```
   x2   0
   x3   0
   x4   0


c =
       x1  x2  x3  x4
   y1   1   0   0   0
   y2   0   0   1   0


d =
       u1
   y1   0
   y2   0

Continuous-time model.
[y,t] = impulse(sys);
figure(1)
plot(t,y)
legend('\theta_1','\theta_2')
xlabel('Time, {\it t} (sec)')
ylabel('\theta_1, \theta_2')
title('Initial condition response for configuration 1')
print -depsc2 'figure1.eps'

%
% Now do configuration 2
%
echo on
%
% The A and B matrix for configuration 2
%
A2 = [0 1 0 0;1.13137 0 0 0;0 0 0 1;0.565685 0 0 0]

A2 =

         0    1.0000         0         0
    1.1314         0         0         0
         0         0         0    1.0000
    0.5657         0         0         0

B2 = [0;0.35697;0;3.45720]

B2 =
```

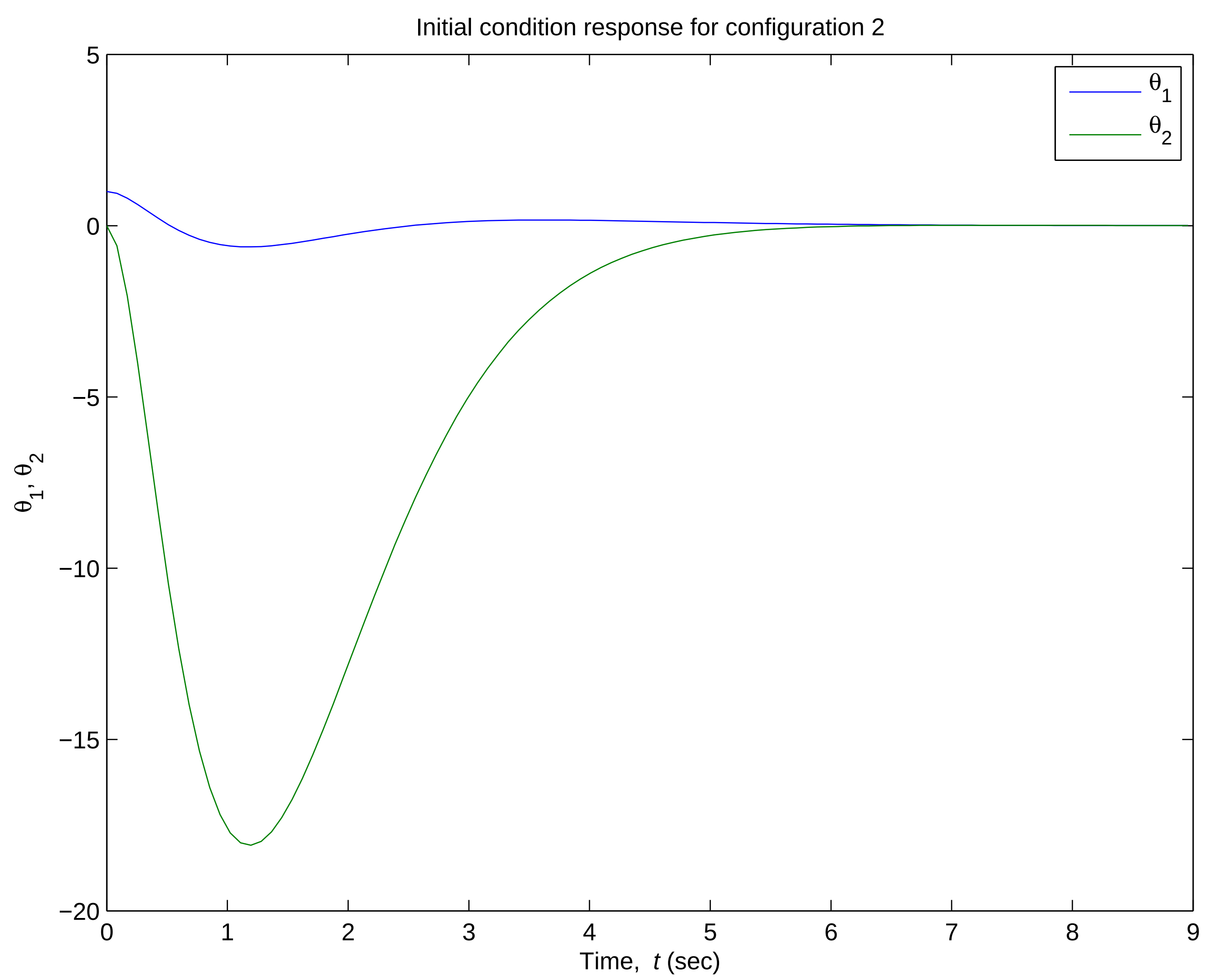
```
         0
    0.3570
         0
    3.4572

%
% (a) The controllability matrix is
%
Mc2 = [B2 A2*B2 A2*A2*B2 A2*A2*A2*B2]

Mc2 =

         0    0.3570         0    0.4039
    0.3570         0    0.4039         0
         0    3.4572         0    0.2019
    3.4572         0    0.2019         0

%
% The controllability matrix is full rank:
%
rank(Mc2)

ans =

     4

%
% (b) The eigenvalues and vectors are:
%
[v,d]=eig(A2);d=diag(d);
%
% The eigenvalues:
%
d

d =

         0
         0
    1.0637
   -1.0637

%
% The eigenvectors:
%
v
```

```
v =

         0         0    0.6127   -0.6127
         0         0    0.6517    0.6517
    1.0000   -1.0000    0.3063   -0.3063
         0    0.0000    0.3258    0.3258

%
% (c) Find the gain K to place all the poles at s=-1.5:
%
K2 = acker(A2,B2,[-1.5,-1.5,-1.5,-1.5])

K2 =

   54.2052   52.0548   -1.3648   -3.6394

%
% Check the results:
%
eig(A2-B2*K2)

ans =

  -1.5003
  -1.5000 + 0.0003i
  -1.5000 - 0.0003i
  -1.4997

%
% (d) Find the homogenous response. There are a number of ways
to do this
% in Matlab. The easiest is that the homogenous response to an
initial
% condition x0 is the same as the impulse response for the
system (A2,x0,C).I
% will find and plot just the position variables, x1 and x3:
%
C = [1 0 0 0;0 0 1 0]

C =

     1     0     0     0
     0     0     1     0

D = [0;0]
```

```
D =

     0
     0

x0 = [1;0;0;0]

x0 =

     1
     0
     0
     0


sys = ss(A2-B2*K2,x0,C,D)

a =
           x1       x2       x3       x4
   x1       0        1        0        0
   x2  -18.22   -18.58   0.4872    1.299
   x3       0        0        0        1
   x4  -186.8     -180    4.718    12.58


b =
       u1
   x1   1
   x2   0
   x3   0
   x4   0


c =
       x1  x2  x3  x4
   y1   1   0   0   0
   y2   0   0   1   0


d =
       u1
   y1   0
   y2   0

Continuous-time model.
```

```
[y,t] = impulse(sys);
figure(2)
plot(t,y)
legend('\theta_1','\theta_2')
xlabel('Time, {\it t} (sec)')
ylabel('\theta_1, \theta_2')
title('Initial condition response for configuration 2')
print -depsc2 'figure2.eps'
>>
```


Initial condition response for configuration 1
θ_1
θ_2
θ_1, θ_2
Time, *t* (sec)
0
1
2
3
4
5
6
7
8
9
0
5
10
15


Initial condition response for configuration 2
θ1
θ2
5
0
−5
−10
−15
−20
0
1
2
3
4
5
6
7
8
9
Time, t (sec)
θ1, θ2

4. Start with the matrix $\begin{pmatrix} sI-(A-BK) & -B\bar{N} \\ C & 0 \end{pmatrix}$

Multiply on the right by $\begin{pmatrix} I & 0 \\ 0 & \bar{N}^{-1} \end{pmatrix}$ on the right (a column reduction) to obtain the matrix $\begin{pmatrix} sI-(A-BK) & -B \\ C & 0 \end{pmatrix}$.

Multiply again on the right by $\begin{pmatrix} I & 0 \\ K & I \end{pmatrix}$ to obtain $\begin{pmatrix} sI-A & -B \\ C & 0 \end{pmatrix}$

Since the two matrices $\begin{pmatrix} I & 0 \\ 0 & \bar{N}^{-1} \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ K & I \end{pmatrix}$ have full rank,

$$\det \begin{pmatrix} sI-(A-BK) & -B\bar{N} \\ C & 0 \end{pmatrix} = 0 \quad \text{iff}$$

$$\det \begin{pmatrix} sI-A & -B \\ C & 0 \end{pmatrix} = 0$$