16.31 Homework 5

Prof. S. R. Hall

Issued: October 23, 2006
Due: October 30, 2006

Problem 1

In some linear quadratic regulator problems, there is a need to add a control weighting cross terms, so that

$$
\begin{aligned}
J &= \int_0^T \left[x^T(t)Qx(t) + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt && (1) \\
&= \int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt && (2)
\end{aligned}
$$

Use dynamic programming to find the optimal control strategy for this problem. What is the Riccati equation for this problem? What is the optimal state feedback gain matrix, F?

Problem 2

Find the Hamiltonian matrix H for the Riccati equation in Problem 1. To derive the Hamiltonian, take $\lambda = Px$, and find the differential equation for the augmented state vector in the form

$$
\frac{d}{dt} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}
$$

Problem 3

The linear quadratic regulator problem seeks to find a controller that keeps the state vector close to the zero vector, using the least amount of control. In the linear quadratic tracking problem, the goal is to keep the state close to a reference vector, $x_r(t)$. The problem is to minimize the cost

$$
J = \int_0^T \left[\{x(t) - x_r(t)\}^T Q \{x(t) - x_r(t)\}^T + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt \qquad (3)
$$

Use dynamic programming to find the optimal control strategy for this problem. What is the optimal control, $u(t)$, as a function of time? Hint: Consider a cost-to-go function of the form

$$
J^*(x(t), t) = x^T(t)P(t)x(t) + q^T(t)x(t) + r(t) \qquad (4)
$$

where $P(t)$ is an $n \times n$ matrix, $q(t)$ is a $n \times 1$ vector, and $r(t)$ is a scalar. To solve the problem, you must derive the differential equations for P (the Riccati equation), q, and r.

In practice, this approach to finding a control law that tracks a changing reference is not practical for real-life control systems. Can you explain why?

Problem 4

For the cost function of Problem 3, find the optimal control, $u(t)$, using Lagrange multiplier methods. You should find that the optimal control depends on the Lagrange multiplier $\lambda(t)$. Find the coupled differential equations for $x(t)$ and $\lambda(t)$. Show that the solution for $\lambda(t)$ can be expressed as

$$\lambda(t) = P(t)x(t) + \lambda_r(t)$$

and find the differential equations for $P(t)$ and $\lambda_r(t)$. Show that the solution you found is equivalent to the solution for Problem 3.