16.31 Homework 5 — Solution

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Problem 1

In some linear quadratic regulator problems, there is a need to add a control weighting cross term, so that

$$J = \int_0^T \left[x^T(t)Qx(t) + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt \tag{1}$$

$$= \int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{2}$$

Use dynamic programming to find the optimal control strategy for this problem. What is the Riccati equation for this problem? What is the optimal state feedback gain matrix, F?

Solution: The cost is given by

$$J^*(x(t),t) = \min_{u(\cdot)} \int_t^T \left[x^T(\tau)Qx(\tau) + 2x^T(\tau)Nu(\tau) + u^T(\tau)Ru(\tau)\right] d\tau$$

$$\approx \min_{u(t)} \left\{ \left[x^T(t)Qx(t) + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt + J^*(x(t+dt), t+dt) \right\} \tag{3}$$

where the approximation sign is used just to indicate that the result is correct to first order in dt. As $dt \to 0$, the result is an equality. The optimal cost-to-go is assumed to be of the form

$$J^*(x(t),t) = x^T(t)P(t)x(t)$$

Therefore, to first order in dt,

$$\begin{aligned} J^*(x(t+dt), t+dt) &\approx \{x(t) + \dot{x}(t)dt\}^T \left\{P(t) + \dot{P}(t)dt\right\} \{x(t) + \dot{x}(t)dt\} \\ &\approx x(t)^T P(t)x(t) + \left[\dot{x}(t)^T P(t)x(t) + x(t)^T P(t)\dot{x}(t) + x(t)^T \dot{P}(t)x(t)\right] dt \\ &= x^T Px + \left[\{Ax + Bu\}^T Px + x^T P \{Ax + Bu\} + x^T \dot{P}x\right] dt \end{aligned} \tag{4}$$

where I've dropped the time argument from all variables when the argument is clear from context. Combining equations (3) and (4),

$$\begin{aligned} J^*(x(t),t) \approx\ & x^T Px + \min_u \Big[x^T Qx + 2x^T Nu + u^T Ru \\ & + \{Ax + Bu\}^T Px + x^T P\{Ax + Bu\} + x^T \dot{P}x\Big] dt \end{aligned} \tag{5}$$

But we already assumed that $J^*(x(t),t) = x^T(t)P(t)x(t)$. Therefore, we must have that

$$\begin{aligned} 0 = \ & \min_u \Big[x^T Qx + 2x^T Nu + u^T Ru \\ & + \{Ax + Bu\}^T Px + x^T P\{Ax + Bu\} + x^T \dot{P}x\Big] dt \end{aligned} \tag{6}$$

To find the minimum of the term in brackets, set its derivative to zero. Then

$$\frac{d}{du}[\cdot] = 2N^T x + 2Ru + 2B^T P x = 0 \tag{7}$$

Solving for u,

$$u(t) = -R^{-1}\left[N^T + B^T P(t)\right] x(t) \tag{8}$$

Therefore, the optimal gain matrix $F(t)$ is

$$F(t) = R^{-1}\left[N^T + B^T P(t)\right] \tag{9}$$

We still have to find $P(t)$ such that equation (6) is satisfied. To do this, plug into the bracked term equation (8) for $u(t)$. Then

$$\begin{aligned} 0 \;=\;& x^T Q x - 2x^T N R^{-1}\left[N^T + B^T P\right] x + \; x^T\left[N + PB\right] R^{-1}\left[N^T + B^T P\right] x \\ &+ x^T\left\{A^T - \left[N + PB\right] R^{-1} B^T\right\} P x \\ &+ x^T P\left\{A - BR^{-1}\left[N^T + B^T P\right]\right\} x + x^T \dot{P} x \end{aligned} \tag{10}$$

I need to make the second term symmetric. The term is a scalar, so its transpose is the same as the scalar itself. So

$$\begin{aligned} 0 \;=\;& x^T Q x - x^T N R^{-1}\left[N^T + B^T P\right] x - x^T\left[N + PB\right] R^{-1} N^T x \\ &+ x^T\left[N + PB\right] R^{-1}\left[N^T + B^T P\right] x \\ &+ x^T\left\{A^T - \left[N + PB\right] R^{-1} B^T\right\} P x \\ &+ x^T P\left\{A - BR^{-1}\left[N^T + B^T P\right]\right\} x + x^T \dot{P} x \end{aligned} \tag{11}$$

All the terms are quadratic in x, and the equation must hold for *every* x. Therefore, the sum of the matrices multiplying x^T and x must be zero:

$$\begin{aligned} 0 \;=\;& Q - N R^{-1}\left[N^T + B^T P\right] - \left[N + PB\right] R^{-1} N^T \\ &+ \left[N + PB\right] R^{-1}\left[N^T + B^T P\right] \\ &+ \left\{A^T - \left[N + PB\right] R^{-1} B^T\right\} P + \; P\left\{A - BR^{-1}\left[N^T + B^T P\right]\right\} + \dot{P} \end{aligned} \tag{12}$$

(Actually, the *symmetric* part of the sum must be zero. But since the sum is symmetric, the sum is zero. This is why we "symmetrized" the expression.) Multiplying out the terms and simplifying,

$$-\dot{P} = \left(A - BR^{-1}N^T\right)^T P + P\left(A - BR^{-1}N^T\right) + Q - NR^{-1}N^T - PBR^{-1}B^T P \tag{13}$$

Problem 2

Find the Hamiltonian matrix H for the Riccati equation in Problem 1. To derive the Hamiltonian, take $\lambda = Px$, and find the differential equation for the augmented state vector in the form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

Solution: To derive the Hamiltonian, take $\lambda = Px$. Then

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax - BR^{-1}\left[N^T + B^T P\right]x \\ &= \left[A - BR^{-1}N^T\right]x - BR^{-1}B^T\lambda \end{aligned} \tag{14}$$

$$\begin{aligned}\dot{\lambda} &= P\dot{x} + \dot{P}x \\ &= PAx + PBu \\ &\quad - \left\{\left(A - BR^{-1}N^T\right)^T P + P\left(A - BR^{-1}N^T\right) + Q - NR^{-1}N^T - PBR^{-1}B^T P\right\}x \\ &= PAx - PBR^{-1}\left[N^T + B^T P\right]x \\ &\quad - \left\{\left(A - BR^{-1}N^T\right)^T P + P\left(A - BR^{-1}N^T\right) + Q - NR^{-1}N^T - PBR^{-1}B^T P\right\}x \\ &= -(Q - NR^{-1}N^T)x - \left(A - BR^{-1}N^T\right)^T Px \\ &= -(Q - NR^{-1}N^T)x - \left(A - BR^{-1}N^T\right)^T \lambda \end{aligned} \tag{15}$$

Therefore,

$$\frac{d}{dt}\left\{\begin{array}{c} x \\ \lambda \end{array}\right\} = \left[\begin{array}{cc} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -\left(A - BR^{-1}N^T\right)^T \end{array}\right]\left\{\begin{array}{c} x \\ \lambda \end{array}\right\} \tag{16}$$

The Hamiltonian matrix is then

$$H = \left[\begin{array}{cc} A - BR^{-1}N^T & -BR^{-1}B^T \\ -Q + NR^{-1}N^T & -\left(A - BR^{-1}N^T\right)^T \end{array}\right] \tag{17}$$

Problem 3

The linear quadratic regulator problem seeks to find a controller that keeps the state vector close to the zero vector, using the least amount of control. In the linear quadratic tracking problem, the goal is to keep the state close to a reference vector, $x_r(t)$. The problem is to minimize the cost

$$J = \int_0^T \left[\{x(t) - x_r(t)\}^T Q \{x(t) - x_r(t)\} + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt \tag{18}$$

Use dynamic programming to find the optimal control strategy for this problem. What is the optimal control, $u(t)$, as a function of time? Hint: Consider a cost-to-go function of the form

$$J^*(x(t), t) = x^T(t)P(t)x(t) + q^T(t)x(t) + r(t) \tag{19}$$

where $P(t)$ is an $n \times n$ matrix, $q(t)$ is a $n \times 1$ vector, and $r(t)$ is a scalar. To solve the problem, you must derive the differential equations for P (the Riccati equation), q, and r.

In practice, this approach to finding a control law that tracks a changing reference is not practical for real-life control systems. Can you explain why?

Solution: The cost-to-go function is

$$J^*(x(t), t) = \min_{u(\cdot)} \int_t^T \left[\{x(t) - x_r(t)\}^T Q \{x(t) - x_r(t)\} + 2x^T(t)Nu(t) + u^T(t)Ru(t)\right] dt \tag{20}$$

We assume a cost-to-go function of the form

$$J^*(x(t),t) = x^T(t)P(t)x(t) + 2q^T(t)x(t) + r(t) \tag{21}$$

where $P(t)$ is an $n \times n$ matrix, $q(t)$ is a $n \times 1$ vector, and $r(t)$ is a scalar. **Note: The original problem statement did not have the factor of 2 in front of q. Putting the factor there simplifies the math a little later on, but is not necessary.** The dynamic programming recursion is then

$$\begin{aligned} J^*(x(t),t) \;\approx\; \min_{u(t)} \Big\{ & \Big[\{x(t)-x_r(t)\}^T Q \{x(t)-x_r(t)\}^T + 2x^T(t)Nu(t) + \; u^T(t)R(t)u(t) \Big] dt \\ & + J^*(x(t+dt),t+dt) \Big\} \end{aligned} \tag{22}$$

As in Problem 1, to first order in dt,

$$\begin{aligned} J^*(x(t+dt),t+dt) \;&\approx\; x(t)^T P(t)x(t) + 2q(t)^T x(t) + r(t) \\ &\quad + \Big[\dot{x}^T(t)P(t)x(t) + x^T(t)P(t)\dot{x}(t) + x(t)^T \dot{P}(t)x(t) \\ &\qquad + 2\dot{q}^T(t)x(t) + 2q^T(t)\dot{x}(t) + \dot{r}(t)dt \Big] \\ &= x^T Px + 2q^T x + r \\ &\quad + \Big[\{Ax+Bu\}^T Px + x^T P\{Ax+Bu\} + x^T \dot{P}x \\ &\qquad + 2\dot{q}^T x + 2q^T(Ax+Bu) + \dot{r} \Big] dt \end{aligned} \tag{23}$$

Combining (20), (21), and (22), we have

$$\begin{aligned} 0 \;=\; \min_u \Big[& \{x-x_r\}^T Q \{x-x_r\} + 2x^T(t)Nu(t) + u^T Ru \\ & + \{Ax+Bu\}^T Px + x^T P\{Ax+Bu\} + x^T \dot{P}x \\ & + 2\dot{q}^T x + 2q^T(Ax+Bu) + \dot{r} \Big] \end{aligned} \tag{24}$$

The minimum occurs for

$$\begin{aligned} \frac{d}{du}[\cdot] &= 0 \\ &= 2N^T x + 2Ru + 2B^T Px + 2B^T q \end{aligned} \tag{25}$$

Solving for u,

$$u(t) = -R^{-1}\left[B^T Px(t) + N^T x(t) + B^T q(t)\right] \tag{26}$$

To complete the problem, we must find $P(t)$ and $q(t)$. (We must also find $r(t)$ if we care about calculating the cost.) To find these quantities, plug (25) back into (23). Then

$$
\begin{aligned}
0 &= \{x - x_r\}^T Q \{x - x_r\} + 2x^T N u + u^T R u \\
&\quad + \{Ax + Bu\}^T P x + x^T P \{Ax + Bu\} + x^T \dot{P} x \\
&\quad + 2\dot{q}^T x + 2q^T (Ax + Bu) + \dot{r} \\
&= \{x - x_r\}^T Q \{x - x_r\}^T - 2x^T N R^{-1} \left[B^T P x + N^T x + B^T q\right] \\
&\quad + \left[B^T P x + N^T x + B^T q\right]^T R^{-1} \left[B^T P x + N^T x + B^T q\right] \\
&\quad + \left\{Ax - B R^{-1} \left[B^T P x + N^T x + B^T q\right]\right\}^T P x \\
&\quad + x^T P \left\{Ax - B R^{-1} \left[B^T P x + N^T x + B^T q\right]\right\} \\
&\quad + x^T \dot{P} x + 2\dot{q}^T x + 2q^T (Ax - B R^{-1} \left[B^T P x + N^T x + B^T q\right]) + \dot{r} \\
&= x^T M_1 x + x^T M_2 + M_3 \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \dot{P} + \left(A - B R^{-1} N^T\right)^T P + P \left(A - B R^{-1} N^T\right) + Q - N R^{-1} N^T - P B R^{-1} B^T P \\
M_2 &= 2\dot{q} - 2Q x_r - 2 P B R^{-1} B^T q + 2A^T q - 2 N R^{-1} B^T q \\
M_3 &= \dot{r} + x_r^T Q x_r - q^T B R^{-1} B^T q
\end{aligned}
$$

Since equation (27) is true for all x, we must have that $M_1 = 0$, $M_2 = 0$, and $M_3 = 0$. Therefore,

$$
\begin{aligned}
-\dot{P} &= \left(A - B R^{-1} N^T\right)^T P + P \left(A - B R^{-1} N^T\right) + Q - N R^{-1} N^T - P B R^{-1} B^T P \qquad (28) \\
-\dot{q} &= \left[A^T - N R^{-1} B^T - P B R^{-1} B^T\right] q - Q x_r \qquad (29) \\
-\dot{r} &= x_r^T Q x_r - q^T B R^{-1} B^T q \qquad (30)
\end{aligned}
$$

For our problem, the boundary conditions are

$$
\begin{aligned}
P(T) &= 0 \qquad (31) \\
q(T) &= 0 \qquad (32) \\
r(T) &= 0 \qquad (33)
\end{aligned}
$$

This approach to tracking control is usually impractical, because we usually don't know the signal we want to track (x_r) in advance. In practice, we must treat the reference signal as a stochastic process, and use LQG to find the optimal control, based only on past values of the reference.

Problem 4

For the cost function of Problem 3, find the optimal control, $u(t)$, using Lagrange multiplier methods. You should find that the optimal control depends on the Lagrange multiplier $\lambda(t)$. Find the coupled differential equations for $x(t)$ and $\lambda(t)$. Show that the solution for $\lambda(t)$ can be expressed as

$$\lambda(t) = P(t)x(t) + \lambda_r(t)$$

and find the differential equations for $P(t)$ and $\lambda_r(t)$. Show that the solution you found is equivalent to the solution for Problem 3.

Solution: Augment the cost with a Lagrange multiplier, so that

$$J = \int_0^T \left[\{x(t) - x_r(t)\}^T Q \{x(t) - x_r(t)\} + 2x^T(t)Nu(t) + u^T(t)Ru(t) + 2\lambda^T(Ax(t) + Bu(t) - \dot{x}) \right] dt \tag{34}$$

Do integration by parts (ignoring boundary conditions) to get

$$J = \int_0^T \left[\{x - x_r\}^T Q \{x - x_r\} + 2x^T Nu + u^T Ru + 2\lambda^T(Ax + Bu) + 2\dot{\lambda}^T x \right] dt \tag{35}$$

Take the variation to obtain

$$\delta J = \int_0^T \left[2\delta x^T Q \{x - x_r\} + 2\delta x^T Nu + 2\delta u^T N^T x + 2\delta u^T Ru + 2\delta x^T A^T \lambda + 2\delta u^T B^T \lambda + 2\delta x^T \dot{\lambda} \right] dt \tag{36}$$

So that variation δx doesn't matter, choose λ so that

$$Q\{x - x_r\} + Nu + A^T\lambda + \dot{\lambda} = 0$$

or

$$\dot{\lambda} = -A^T\lambda - Q\{x - x_r\} - Nu$$

Then the cost becomes

$$\delta J = 2\int_0^T \left[\delta u^T N^T x + \delta u^T Ru + 2\delta u^T B^T \lambda \right] dt \tag{37}$$

So that the cost is stationary, we must have that the integrand is zero, so that

$$N^T x + Ru + B^T\lambda = 0$$

or

$$u = -R^{-1}B^T\lambda - R^{-1}N^T x$$

Now, assume that λ has the form

$$\lambda(t) = P(t)x(t) + \lambda_r(t)$$

Then we must have that

$$\begin{aligned} \dot{\lambda} &= \dot{P}x + P\dot{x} + \dot{\lambda}_r \\ &= -A^T\lambda - Q\{x - x_r\} - Nu \end{aligned}$$

or

$$0 = \dot{P}x + P\dot{x} + \dot{\lambda}_r + A^T\lambda + Q\{x - x_r\} + Nu$$

Furthermore, we must that that

$$\begin{aligned} u &= -R^{-1}B^T\lambda - R^{-1}N^Tx \\ &= -R^{-1}\left(B^TPx + N^Tx + B^T\lambda_r\right) \end{aligned}$$

Expanding $\dot{x}$ and substituting in the expression for u, we have that

$$\begin{aligned} 0 &= \dot{P}x + P(Ax + Bu) + \dot{\lambda}_r + A^T\lambda + Q\{x - x_r\} + Nu \\ &= \dot{P}x + P\left(Ax - BR^{-1}\left(B^TPx + N^Tx + B^T\lambda_r\right)\right) + \dot{\lambda}_r + A^T\left(Px + \lambda_r\right) \\ &\qquad + Q\{x - x_r\} - NR^{-1}\left(B^TPx + N^Tx + B^T\lambda_r\right) \\ &= \left(\dot{P} + PA + A^TP - PBR^{-1}B^TP - PBR^{-1}N^T - NR^{-1}B^TP + Q - NR^{-1}N^T\right)x \\ &\qquad + PB^T\lambda_r + \dot{\lambda}_r - Qx_r - NR^{-1}B^T\lambda_r \end{aligned}$$

This equation is satisfied if

$$-\dot{P} = \left(A - BR^{-1}N^T\right)^T P + P\left(A - BR^{-1}N^T\right) + Q - NR^{-1}N^T - PBR^{-1}B^TP \quad (38)$$

$$-\dot{\lambda}_r = \left[A^T - NR^{-1}B^T - PBR^{-1}B^T\right]\lambda_r - Qx_r \quad (39)$$

This are precisely the same equations we had earlier, if we identify q and λ_r.