16.31 Problem Set 6
Solution

Problem 1

The matrix XKY is given by

$$[XKY]_{ij} = \sum_k \sum_l X_{ik} K_{kl} Y_{lj}$$

The trace is then given by

$$\begin{aligned} tr[XKY] &= \sum_i [XKY]_{ii} \\ &= \sum_i \sum_k \sum_l X_{ik} K_{kl} Y_{li} \end{aligned}$$

The derivative is given by

$$\left\{ \frac{d}{dK} tr[XKY] \right\}_{mn} = \frac{d}{dK_{mn}} tr[XKY]$$

$$= \frac{d}{dK_{mn}} \sum_i \sum_k \sum_l X_{ik} K_{kl} Y_{li}$$

$$= \sum_i \sum_k \sum_l X_{ik} \underbrace{\frac{dK_{kl}}{dK_{mn}}}_{= \begin{cases} 1 & \text{if } k=m,\ l=n \\ 0 & \text{else} \end{cases}} Y_{li}$$

$$= \sum_i X_{im} Y_{ni}$$

$$= \sum_i Y_{ni} X_{im} = [YX]_{nm}$$

$$\Rightarrow \frac{d}{dK} tr[XKY] = (YX)^T = X^T Y^T$$

Problem 2

$$\dot{\Sigma} = (A - KC)\Sigma + \Sigma(A-KC)^T + GWG^T + KVK^T$$

$K = K^* + \delta K$, where $K^* = \Sigma C^T V^{-1}$

$$\Rightarrow \dot{\Sigma} = (A - [\Sigma C^T V^{-1} C + \delta K C])\Sigma + \Sigma(\cdot)^T$$
$$+ GWG^T + (\Sigma C^T V^{-1} + \delta K) V (\cdot)^T$$

$$= A\Sigma + \Sigma A^T + GWG^T - \Sigma C^T V^{-1} C \Sigma$$
$$- \delta K C \Sigma - \Sigma C^T \delta K^T + \delta K C \Sigma + \Sigma C^T \delta K^T$$
$$+ \delta K V \delta K^T$$

$$= \underbrace{\left\{ A\Sigma + \Sigma A^T + GWG^T - \Sigma C^T V^{-1} C \Sigma \right\}}_{\substack{\text{what we claimed was the} \\ \text{minimum} \\ = (\dot{\Sigma})^*}} + \underbrace{\delta K V \delta K^T}_{\substack{\text{positive} \\ \text{definite matrix}}}$$

So any K other than $K = \Sigma C^T V^{-1}$ gives a more positive rate of change.

Problem 3

Form the Hamiltonian

$$H = \begin{bmatrix} A^T & -C^T V^{-1} C \\ -GWG^T & -A \end{bmatrix}$$

Find the eigenvalues and eigenvectors of H. Form a matrix, $\mathcal{V}$, which has columns that are the eigenvectors corresponding to stable eigenvectors. Partition $\mathcal{V}$ into two $n \times n$ matrices, X and Y, so that

$$\mathcal{V} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then $\quad \Sigma = Y X^{-1}$