16.31 Homework 7

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1. Consider a general state-space system with state equations

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

and transfer function $G(s) = C(sI - A)^{-1}B + D$. For simplicity, we will write the system in the shorthand form

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Assume that the transfer function matrix $G(s)$ is square. Find the inverse transfer function matrix, $G^{-1}(s)$, in state-space form, expressed in the shorthand form above. Hint: First solve for u in terms of y, and then find the new state equation. What conditions are required for there to be an inverse in state-space form?

2. For a transfer function $G(s)$ defined as in Part 1, the transfer function $G^{\sim}(s)$ is defined as

$$G^{\sim}(s) = G^T(-s)$$

This "para-Hermitian" is useful because for $s = j\omega$, $G^{\sim}(j\omega) = G^*(j\omega)$, the complex conjugate transpose of the transfer function, Find the (shorthand) state-space form for $G^{\sim}(s)$.

3. For a system driven by white noise, with white measurement noise,

$$\begin{aligned}\dot{x} &= Ax + Gw \\ y &= Cx + v\end{aligned}$$

the power spectral density of y is given by

$$\Phi_{yy}(\omega) = G(s)WG^{\sim}(s) + V$$

where the right hand side is evaluated at $s = j\omega$, and W and V are the white noise intensities. Find a state-space description in shorthand form for $H(s) \equiv G(s)WG^{\sim}(s) + V$.

4. Using the results of Part 1, find a state-space description for $H^{-1}(s)$. Do you recognize the A matrix of the result?

5. The trickiest part remaining is to factor $H^{-1}(s)$. To do this, transform the state using a state transformation matrix of the form

$$T = \begin{bmatrix} I & \pm\Sigma \\ 0 & I \end{bmatrix} \qquad \text{or} \qquad T = \begin{bmatrix} I & 0 \\ \pm\Sigma & I \end{bmatrix}$$

The "right" choice of T will be such that that the transformed A matrix will be triangular — it will have a zero in one of the off-diagonal entries. What is the condition of Σ that is required to make the A matrix triangular? Hint: We're solving the steady-state Kalman filter problem!

Once you've found the right T, you should be able to factor $H(s)^{-1}$ as

$$H(s)^{-1} = F^{\sim}(s)V^{-1}F(s)$$

What is $F(s)$ in state-space form? Hint: Assume a state-space description for $F(s)$ (including a D term), and form the product on the right hand side above. Then match terms to find $F(s)$. You may find that you have to do a minor state transformation (such as changing the sign of one of the states) to make things work out.

6. For this $F(s)$, we have that

$$H^{-1}(s) = F^{\sim}(s)V^{-1}F(s)$$

and therefore

$$F(s)H(s)F^{\sim}(s) = V$$

Therefore, if we let the input to $F(s)$ be y, and the output be called r, then the spectrum of r is

$$\Phi_{rr}(\omega) = F(j\omega)\Phi_{yy}(\omega)F^{\sim}(j\omega) = V$$

That is, F is a filter that "whitens" the signal y, and produces a white noise signal as its output. Show that r is in fact the Kalman filter residual!