16.31 Homework 7 Solution

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1. Consider a general state-space system with state equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and transfer function $G(s) = C(sI - A)^{-1}B + D$. For simplicity, we will write the system in the shorthand form

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Assume that the transfer function matrix $G(s)$ is square. Find the inverse transfer function matrix, $G^{-1}(s)$, in state-space form, expressed in the shorthand form above. Hint: First solve for u in terms of y, and then find the new state equation. What conditions are required for there to be an inverse in state-space form?

Solution: We can solve for u in terms of y, etc., as

$$u = -D^{-1}Cx + D^{-1}y$$

Then

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= (A - BD^{-1}C)x + BD^{-1}y \end{aligned}$$

Therefore,

$$G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

2. For a transfer function $G(s)$ defined as in Part 1, the transfer function $G^{\sim}(s)$ is defined as

$$G^{\sim}(s) = G^T(-s)$$

This "para-Hermitian" is useful because for $s = j\omega$, $G^{\sim}(j\omega) = G^*(j\omega)$, the complex conjugate transpose of the transfer function, Find the (shorthand) state-space form for $G^{\sim}(s)$.

Solution: We have that

$$\begin{aligned} G^T(-s) &= \left[C\left(-sI - A\right)^{-1}B + D\right]^T \\ &= -B^T\left(sI + A^T\right)^{-1}C^T + D^T \end{aligned}$$

Therefore,

$$G^{\sim}(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$$

3. For a system driven by white noise, with white measurement noise,

$$\dot{x} = Ax + Gw$$
$$y = Cx + v$$

the power spectral density of y is given by

$$\Phi_{yy}(\omega) = G(s)WG^{\sim}(s) + V$$

where the right hand side is evaluated at $s = j\omega$, and W and V are the white noise intensities. Find a state-space description in shorthand form for $H(s) \equiv G(s)WG^{\sim}(s) + V$.

Solution: For this problem,

$$G(s) = \left[\begin{array}{c|c} A & G \\ \hline C & 0 \end{array}\right], \qquad G^{\sim}(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline G^T & 0 \end{array}\right]$$

Then $H(s)$ can be found by writing out the state-space descriptions of $G(s)$ and $G^{\sim}(s)$, forming a larger state vector consisting of the states of both systems, and then writing the result in shorthand form. The result is

$$H(s) = \left[\begin{array}{cc|c} A & GWG^T & 0 \\ 0 & -A^T & -C^T \\ \hline C & 0 & V \end{array}\right]$$

4. Using the results of Part 1, find a state-space description for $H^{-1}(s)$. Do you recognize the A matrix of the result?

Solution: Simplify apply the results of Part 1 directly to obtain

$$H^{-1}(s) = \left[\begin{array}{cc|c} A & GWG^T & 0 \\ C^TV^{-1}C & -A^T & -C^TV^{-1} \\ \hline V^{-1}C & 0 & V^{-1} \end{array}\right] = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$$

The matrix $\overline{A}$ is the Hamiltonian for the Kalman filter problem!

5. The trickiest part remaining is to factor $H(s)^{-1}$. To do this, transform the state using a state transformation matrix of the form

$$T = \begin{bmatrix} I & \pm\Sigma \\ 0 & I \end{bmatrix} \qquad \text{or} \qquad T = \begin{bmatrix} I & 0 \\ \pm\Sigma & I \end{bmatrix}$$

The "right" choice of T will be such that that the transformed A matrix will be triangular — it will have a zero in one of the off-diagonal entries. You will know that you've picked the right transformation. What is the condition of Σ that is required to make the A matrix triangular? Hint: We're solving the steady-state Kalman filter problem!

Once you've found the right T, you should be able to factor $H(s)^{-1}$ as

$$H(s)^{-1} = F^{\sim}(s)V^{-1}F(s)$$

What is $F(s)$ in state-space form? Hint: Assume a state-space description for $F(s)$ (including a D term), and form the product on the right hand side above. Then match terms to find $F(s)$.

Solution: Take

$$T = \begin{bmatrix} I & -\Sigma \\ 0 & I \end{bmatrix}$$

Then

$$H^{-1}(s) = \left[\begin{array}{c|c} T\overline{A}T^{-1} & T\overline{B} \\ \hline \overline{C}T^{-1} & \overline{D} \end{array}\right]$$

Note that

$$T^{-1} = \begin{bmatrix} I & \Sigma \\ 0 & I \end{bmatrix}$$

Doing the (grungy) multiplication yields

$$H^{-1}(s) = \left[\begin{array}{cc|c} A - \Sigma C^T V^{-1} C & A\Sigma + \Sigma A^T + GWG^T - \Sigma C^T V^{-1} C \Sigma & \Sigma C^T V^{-1} \\ C^T V^{-1} C & -A^T + C^T V^{-1} C \Sigma & -C^T V^{-1} \\ \hline V^{-1} C & V^{-1} C \Sigma & V^{-1} \end{array}\right] = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$$

To make $\overline{A}$ triangular, require that

$$0 = A\Sigma + \Sigma A^T + GWG^T - \Sigma C^T V^{-1} C \Sigma$$

This is just the algebraic Riccati equation for the Kalman Filter! To simplify notation, let $L = \Sigma C^T V^{-1}$. Then

$$H^{-1}(s) = \left[\begin{array}{cc|c} A - LC & 0 & L \\ C^T V^{-1} C & -A^T + C^T L^T & -C^T V^{-1} \\ \hline V^{-1} C & L^T & V^{-1} \end{array}\right]$$

It is easily verified that if $F(s)$ is given by

$$F(s) = \left[\begin{array}{c|c} A - LC & L \\ \hline -C & I \end{array}\right]$$

then

$$F^{\sim}(s) V^{-1} F(s) = H^{-1}(s)$$

Note that L is the Kalman filter gain!

6. For this $F(s)$, we have that

$$H^{-1}(s) = F^{\sim}(s) V^{-1} F(s)$$

and therefore

$$F(s) H(s) F^{\sim}(s) = V$$

Therefore, if we let the input to $F(s)$ be y, and the output be called r, then the spectrum of r is

$$\Phi_{rr}(\omega) = F(j\omega)\Phi_{yy}(\omega)F^{\sim}(j\omega) = V$$

That is, F is a filter that "whitens" the signal y, and produces a white noise signal as its output. Show that r is in fact the Kalman filter residual!

Solution: There's not much to do here. The Kalman filter is

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + L(y - C\hat{x}) \\ r &= y - C\hat{x}\end{aligned}$$

which is in fact just $F(s)$ operating on y, as desired.