16.31 Problem Set 7
Solution

Problem 1

See the attached Matlab output, which has on it the resulting compensator.

The gain margin is the inverse of the loop gain magnitude, when the phase is $\pm 180°$. The gain margin is then

$$GM = [0.84,\ 1.15]$$

upward gain margin (1.15)
downward gain margin (0.84)

The phase margin is the difference between the phase of the loop gain and $\pm 180°$ when the magnitude of the loop gain is unity.

$$PM = [-8.1°,\ 19.3°]$$

additional phase lag allowed (19.3°)
additional phase lead allowed. (−8.1°)

Inherently, this is a hard plant to control — it has an unstable pole near a right half plane zero. It is inevitable that the margins will be low.

```
cc=f;
dc=0;

%  Now put back into state space form

[numc,denc]=ss2tf(ac,bc,cc,dc,1)

numc =

         0   82.5288  247.5690


denc =

    1.0000    9.2304  -47.4282
```

$$K(s) = \frac{82.5s + 248}{s^2 + 9.23s - 47.4}$$

```
%  Is it stable?

roots(denc)

ans =

  -12.9055
    3.6750
```

stable pole

unstable pole!

```
%  Now prepare the nyquist plot

num=conv(numg,numc);
den=conv(deng,denc);
w=logspace(-2,2,200);
[mag,phase]=bode(num(3:4),den,w);
[re,im]=nyquist(num,den,w);
hold off
clf
axis([-1.2 0 -0.4 0.4])
plot(re,im)
hold on
plot(re,-im)


%  Find largest circle around -1 point and draw, to see the robustness.

re2=re+1;
mag2=abs(re2+j*im);

r=min(mag2)

r =

    0.1300
```

radius of circle around s=-1 that just touches nyquist plot.

```
th=(0:.002:1)*2*pi;
circle=exp(j*th);
plot(r*circle-1)
diary off
```

```
%  P6-1 Solution

% Define j

j=sqrt(-1);

%  Express G as a transfer function, then put into statespace form

numg=[0 1 -1];
deng=conv([1 -2],[1 3]);
```

} G(s) from problem statement

```
[a,b,c,d]=tf2ss(numg,deng)
```

tf2ss puts G(s) in state-space form, using phase-variable form, but with states in reverse order.

```
a =

    -1     6
     1     0


b =

     1
     0


c =

     1    -1


d =

     0


%  Find the Kalman filter gain

[k,p]=lqe(a,b,c,1,1);
k
```

W V

```
k =

    8.1647
    4.0495
```

Kalman gain

```
%  Find the LQR gain--add a small matrix to c'*c to make positive
%  definite.

[f,p]=lqr(a,b,c'*c+30*eps*eye(2),1);
f
```

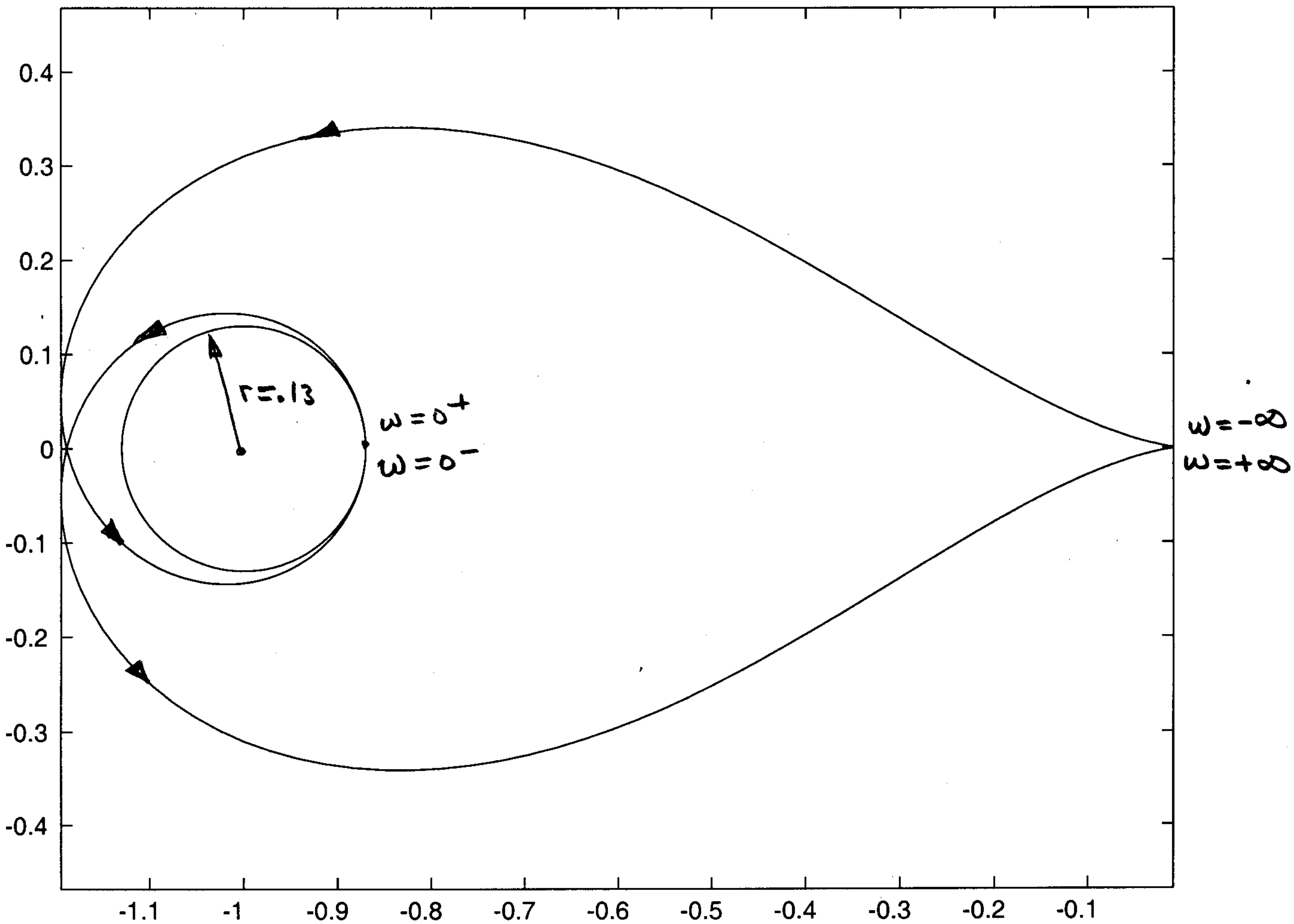
C^TC

```
f =

    4.1152   12.0828
```

LQR gain

```
%  The Compensator is found using the separation principle

ac=a-b*f-k*c;
bc=k;
```

Nyquist plot



Problem 2

The singular values of T and T^* are the square roots of the eigenvalues of T^*T and TT^*, respectively. The maximum singular value is the maximum of these, which will be nonzero, unless T is the zero matrix. So it suffices to show that any nonzero eigenvalue of T^*T is an eigenvalue of TT^*, and vice versa.

So suppose

$$T^*T\underline{p} = \lambda \underline{p}$$

Define $\underline{q} = T\underline{p}$. Then

$$\begin{aligned} TT^*\underline{q} &= TT^*T\underline{p} \\ &= T(\lambda \underline{p}) = \lambda T\underline{p} \\ &= \lambda \underline{q} \end{aligned}$$

So if λ is an eigenvalue of T^*T, it is also an eigenvalue of TT^*. ∎

Problem 3

See the following two pages for the Matlab code and a test run of the code

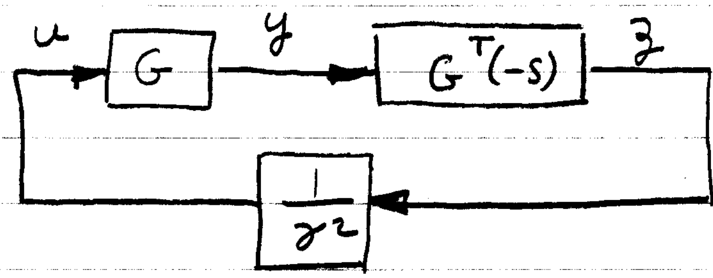
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% ~~P6-3~~ Solution 7.3
%
%%~%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
function y=infnorm(A,B,C)

%
% Initial guess for gamma

gamma=1;
gamma_upper=Inf;
gamma_lower=0;

%
% Iterate until upper bound and lower bound converge

while (gamma_upper-gamma_lower)/gamma_upper > 10*eps |  gamma_upper==Inf

% Form the Hamiltonian matrix
%

H=[   A  B*B'/gamma^2 ;
   -C'*C     -A'       ];

% Find the eigenvalues of H, and decide if any are on jw axis
% This part is tricky, so you may use a different approach.

d=eig(H);

if min(abs(real(d)))/max(abs(d)) < 1e-14
        jw=1;
else
        jw=0;
end

% based on result, move bounds

if jw         % gamma is higher, so move up
        if gamma_upper==Inf
                gamma_lower=gamma;
                gamma=gamma*2;
        else gamma_lower=gamma;
                gamma=(gamma_upper+gamma_lower)/2;
        end
else         % gamma is lower, so split the difference
        gamma_upper=gamma;
        gamma=(gamma_upper+gamma_lower)/2;
end
end
y=gamma ;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%  P6-3 test script
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear

A=[-0.1 1 ;
   -2 -0.3];

B=[1 2 ;
   3 4];

C=[1  1 ;
   1 -1];

infnorm(A,B,C)
echo off
diary off


ans =

   25.9297
```

$\|G\|_\infty$

Problem 4

The block diagram to check $\|G\|_\infty < \gamma$ is



The equivalent state-space dynamics are

$$\left.\begin{aligned} \dot{x}_1 &= A x_1 + B u \\ y &= C x_1 + D u \end{aligned}\right\} G$$

$$\left.\begin{aligned} \dot{x}_2 &= -A^T x_2 - C^T y \\ z &= B^T x_2 + D^T y \end{aligned}\right\} G^T(-s)$$

$$u = \frac{1}{\gamma^2} z$$

We need to eliminate y, u, and z from these equations:

$$\begin{aligned} y &= C x_1 + D u \\ &= C x_1 + \frac{D}{\gamma^2} z \\ &= C x_1 + \frac{1}{\gamma^2} D \left(B^T x_2 + D^T y \right) \end{aligned}$$

$$\Rightarrow \left(I - \frac{1}{\gamma^2} DD^T\right) y = C x_1 + \frac{1}{\gamma^2} DB^T x_2$$

$$\Rightarrow (\gamma^2 I - DD^T) y = \gamma^2 C x_1 + DB^T x_2$$

$$\Rightarrow y = (\gamma^2 I - DD^T)^{-1}\left[\gamma^2 C x_1 + DB^T x_2\right]$$

Similarly,

$$u = (\gamma^2 I - D^T D)^{-1}\left[B^T x_2 + D^T C x_1\right]$$

Therefore, the state dynamics are

$$\begin{aligned}
\dot{x}_1 &= A x_1 + B u \\
&= \left\{A + B(\gamma^2 I - D^T D)^{-1} D^T C\right\} x_1 \\
&\quad + \left\{ B(\gamma^2 I - D^T D)^{-1} B^T\right\} x_2
\end{aligned}$$

$$\begin{aligned}
\dot{x}_2 &= -A^T x_2 - C^T y \\
&= -\left\{A + B D^T (\gamma^2 I - DD^T)^{-1} C\right\}^T x_2 \\
&\quad - C^T (\gamma^2 I - DD^T)^{-1} \gamma^2 C x_1
\end{aligned}$$

This can be simplified some.

$$\dot{x}_1 = \{A + B\hat{D}D^TC\}x_1 + \{B\hat{D}B^T\}x_2$$

where $\hat{D} = (\gamma^2 I - D^T D)^{-1}$

Also, note that

$$D^T(\gamma^2 I - DD^T)^{-1} = (\gamma^2 I - D^T D)^{-1} D$$
$$= \hat{D} D$$

so
$$\dot{x}_2 = -\{A + B\hat{D}D^TC\}^T x_2$$
$$-\{C^T(\gamma^2 I - DD^T)^{-1}\gamma^2 C\}x_1$$

We would like to express $(\gamma^T I - DD^T)^{-1}$ in terms of $\hat{D}$, although this is not necessary. To do this, note that

$$I = (\gamma^2 I - DD^T)^{-1}(\gamma^2 I - DD^T)$$
$$= (\gamma^2 I - DD^T)^{-1}\gamma^2 - (\gamma^2 I - DD^T)^{-1}DD^T$$

$$\Rightarrow (\gamma^2 I - DD^T)^{-1}\gamma^2 = I + (\gamma^2 I - DD^T)^{-1}DD^T$$

$$= I + D(\gamma^2 I - D^T D)^{-1}D^T$$
$$= I + D\hat{D}D^T$$

Therefore,

$$\dot{x}_2 = -(A + B\hat{D}D^TC)^T x_2 - (C^TC + C^TD\hat{D}D^TC)x_1$$

Therefore, the Hamiltonian is

$$H = \begin{bmatrix} A + B\hat{D}D^TC & +B\hat{D}B^T \\ -C^TC - C^TD\hat{D}DC & -(A + B\hat{D}D^TC)^T \end{bmatrix}$$

$$\hat{D} = (\gamma^2 I - D^TD)^{-1}$$

Having obtained the Hamiltonian, the algorithm is the same, <u>except</u> that we must start with the lower bound

$$\gamma_{lower} = \bar{\sigma}(D)$$

See the Matlab function and test script on the following pages.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%  P6/4 Solution  7.4
%
%%~%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
function y=infnorm2(A,B,C,D)

%
%  Initial guess for gamma

gamma_upper=Inf;
gamma_lower=max(svd(D));
gamma=gamma_lower*2+1;          % Guess bigger than gamma_lower and less than
                                                          % gamma_upper


%
%  Iterate until upper bound and lower bound converge

while (gamma_upper-gamma_lower)/gamma_upper > 10*eps |  gamma_upper==Inf

%  Form the Hamiltonian matrix
%

[m,n]=size(D);
Dh=inv(eye(m)*gamma^2-D'*D);

H=[   A+B*Dh*D'*C              B*Dh*B' ;
   -C'*C-C'*D*Dh*D'*C     -(A+B*Dh*D'*C)'];

%  Find the eigenvalues of H, and decide if any are on jw axis
%  This part is tricky, so you may use a different approach.

d=eig(H);
if min(abs(real(d)))/max(abs(d)) < 1e-14
        jw=1;
else
        jw=0;
end

%  based on result, move bounds

if jw        % gamma is higher, so move up
        if gamma_upper==Inf
                gamma_lower=gamma;
                gamma=gamma*2;
        else gamma_lower=gamma;
                gamma=(gamma_upper+gamma_lower)/2;
        end
else        % gamma is lower, so split the difference
        gamma_upper=gamma;
        gamma=(gamma_upper+gamma_lower)/2;
end
end
y=gamma ;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%  P6-4 test script  7.4
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear

A=[-0.1 1 ;
   -2 -0.3];

B=[1 2 ;
   3 4];

C=[1  1 ;
   1 -1];

D=[20  3 ;
   4  26];

infnorm2(A,B,C,D)

ans =

   43.6829
```

$\|G\|_\infty$