« An Informal Overview of Abstract Interpretation »

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Course 16.399: "Abstract interpretation"
http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/

What is static analysis by abstract interpretation?

Example of static analysis (input)

```
n := n0;

i := n;

while (i <> 0 ) do

      j := 0;

      while (j <> i) do

            j := j + 1

      od;

      i := i - 1

od
```


Example of static analysis (output)

```
{n0>=0}
   n := n0;
{n0=n,n0>=0}
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
{n0=n,i=0,n0>=0}
```

Example of static analysis (safety)

```
{n0>=0}
   n := n0;
{n0=n,n0>=0}
   i := n;
{n0=i,n0=n,n0>=0}
   while (i <> 0 ) do
      {n0=n,i>=1,n0>=i}
         j := 0;
      {n0=n,j=0,i>=1,n0>=i}
         while (j <> i) do
            {n0=n,j>=0,i>=j+1,n0>=i}
               j := j + 1      ⟵  j < n0 so no upper overflow
            {n0=n,j>=1,i>=j,n0>=i}
         od;
      {n0=n,i=j,i>=1,n0>=i}
         i := i - 1            ⟵  i > 0 so no lower overflow
      {i+1=j,n0=n,i>=0,n0>=i+1}
   od
{n0=n,i=0,n0>=0}
```

n0 must be initially nonnegative (otherwise the program does not terminate properly)

Static analysis by abstract interpretation

Verification: define and prove automatically a property of the possible behaviors of a complex computer program;

Abstraction: the reasoning/calculus can be done on an abstraction of these behaviors dealing only with those elements of the behaviors related to the considered property;

Theory: abstract interpretation.

Example of static analysis

Verification: absence of runtime errors;

Abstraction: polyhedral abstraction (affine inequalities);

Theory: abstract interpretation.

Potential impact of runtime errors

– 50% of the security attacks on computer systems are through buffer overruns [1]!

– Embedded computer system crashes easily result from overflows of various kinds.

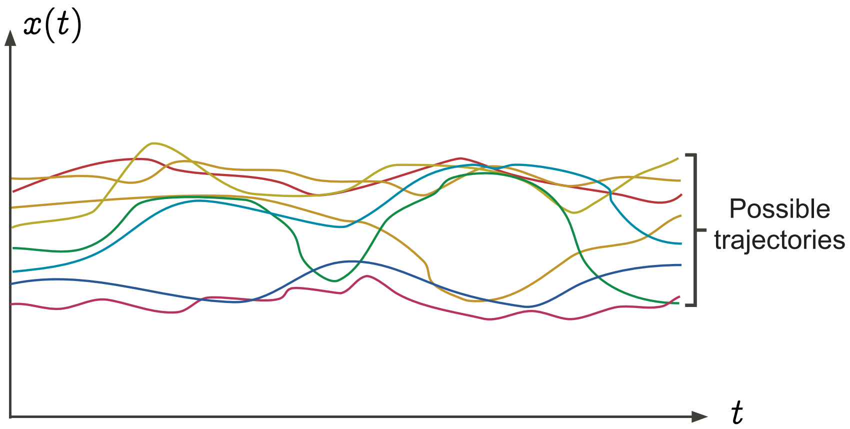
[1] See for example the Microsoft Security Bulletins MS02-065, MS04-011, etc.

A very informal introduction to the principles of abstract interpretation

Semantics

The *concrete semantics* of a program formalizes (is a mathematical model of) the set of all its possible executions in all possible execution environments.

Graphic example: Possible behaviors



Undecidability

- The concrete mathematical semantics of a program is an "infinite" mathematical object, *not computable*;
- All non trivial questions on the concrete program semantics are *undecidable*.

Example: Kurt Gödel argument on termination

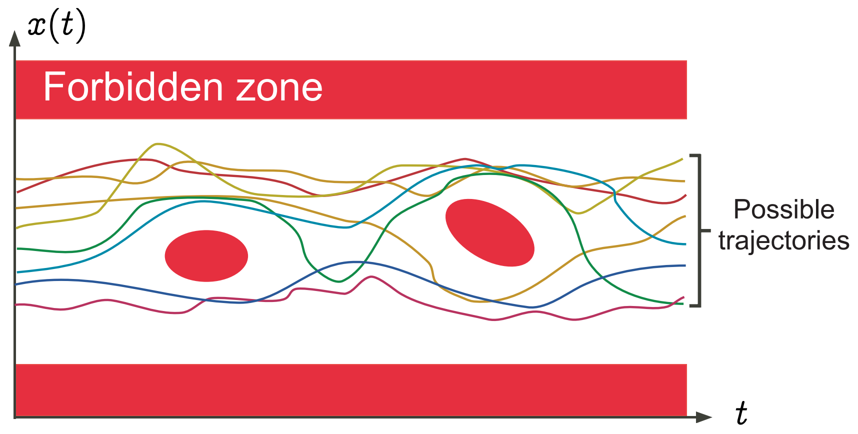
- Assume `termination(P)` would always terminates and returns `true` iff `P` always terminates on all input data;
- The following program yields a contradiction

$$\texttt{P} \equiv \texttt{while termination(P) do skip od.}$$

Graphic example: Safety properties

The *safety properties* of a program express that no possible execution in any possible execution environment can reach an erroneous state.

Graphic example: Safety property



Safety proofs

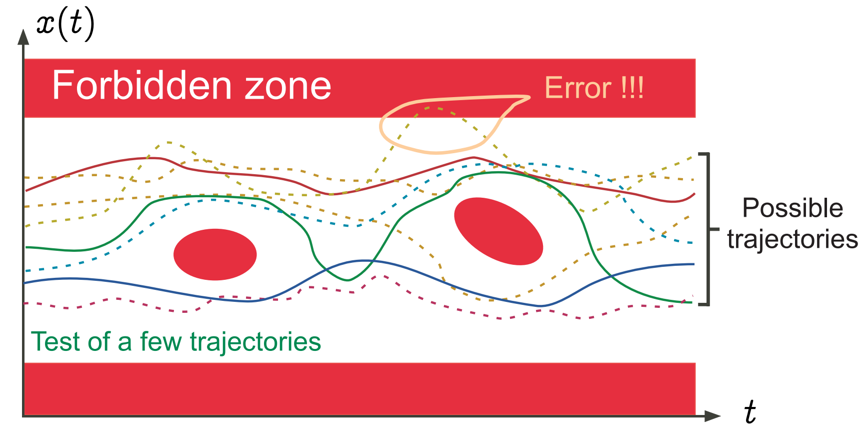
- A safety proof consists in proving that the intersection of the program concrete semantics and the forbidden zone is empty;
- Undecidable problem (the concrete semantics is not computable);
- Impossible to provide completely automatic answers with finite computer resources and neither human interaction nor uncertainty on the answer [2].

[2] e.g. probabilistic answer.

Test/debugging

- consists in considering a subset of the possible executions;
- not a correctness proof;
- absence of coverage is the main problem.

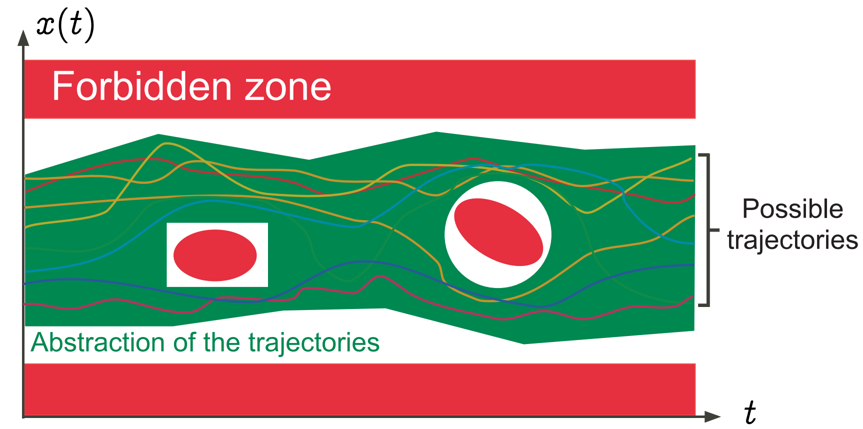
Graphic example: Property test/simulation



Abstract interpretation

- consists in considering an *abstract semantics*, that is to say a superset of the concrete semantics of the program;
- hence the abstract semantics covers all possible concrete cases;
- correct: if the abstract semantics is safe (does not intersect the forbidden zone) then so is the concrete semantics.

Graphic example: Abstract interpretation



Formal methods

Formal methods are abstract interpretations, which differ in the way to obtain the abstract semantics:

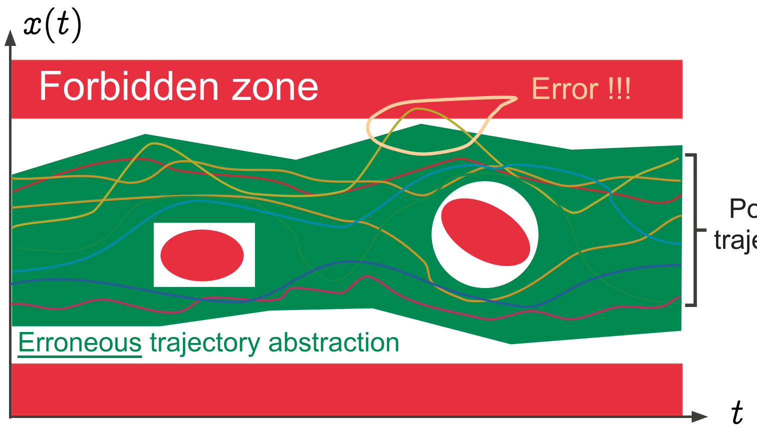
- "*model checking*":
  - the abstract semantics is given manually by the user;
  - in the form of a finitary model of the program execution;
  - can be computed automatically, by techniques relevant to static analysis.

– "*deductive methods*":
  - the abstract semantics is specified by verification conditions;
  - the user must provide the abstract semantics in the form of inductive arguments (e.g. invariants);
  - can be computed automatically by methods relevant to static analysis.
– "*static analysis*": the abstract semantics is computed automatically from the program text according to predefined abstractions (that can sometimes be tailored automatically/manually by the user).

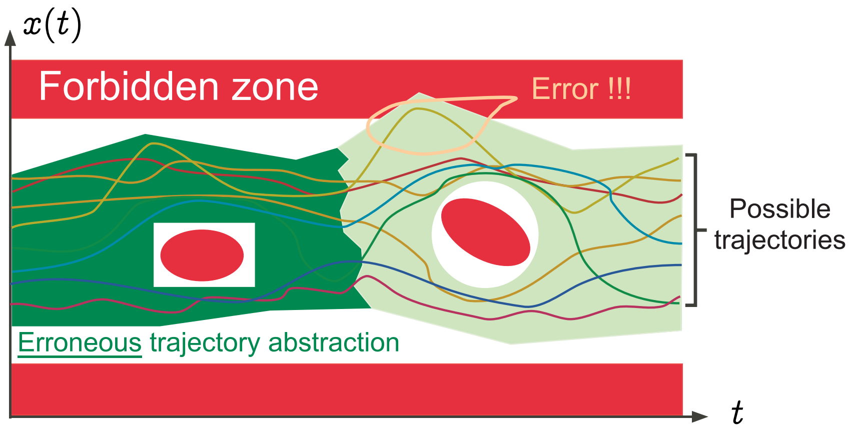
Required properties of the abstract semantics

– sound so that no possible error can be forgotten;
– precise enough (to avoid false alarms);
– as simple/abstract as possible (to avoid combinatorial explosion phenomena).

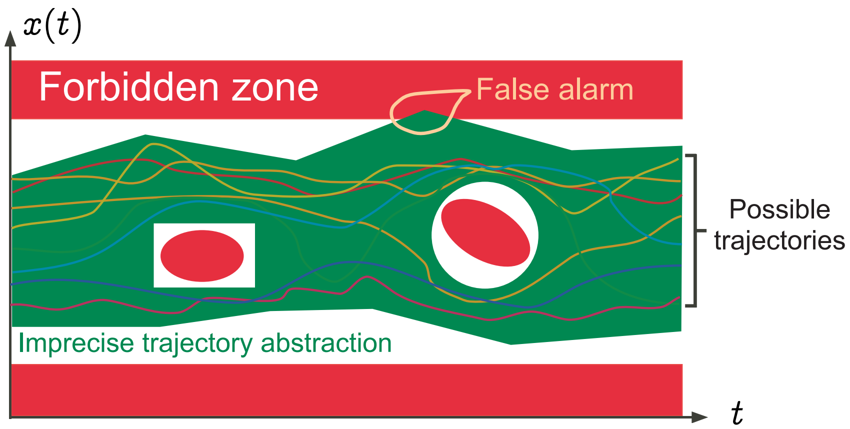
Graphic example: Erroneous abstraction — I



Graphic example: Erroneous abstraction — II



Graphic example: Imprecision ⇒ false alarms

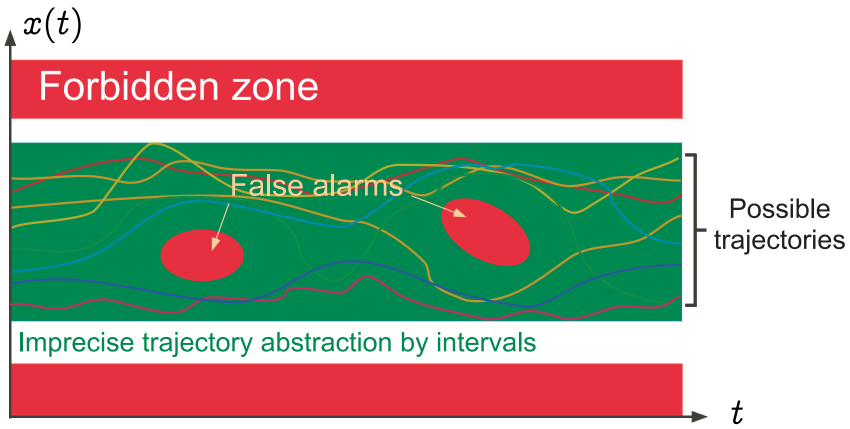


Abstract domains

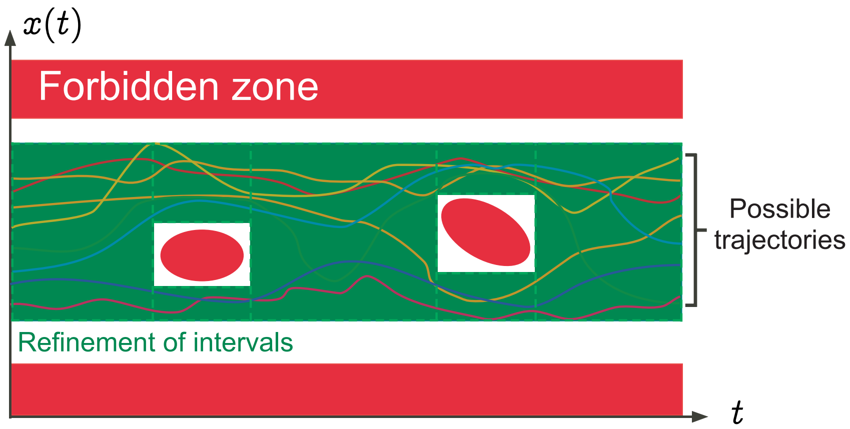
Standard abstractions

- that serve as a basis for the design of static analyzers:
  - abstract program data,
  - abstract program basic operations;
  - abstract program control (iteration, procedure, concurrency, ...);
- can be parametrized to allow for manual adaptation to the application domains.

Graphic example: Standard abstraction by intervals



Graphic example: A more refined abstraction



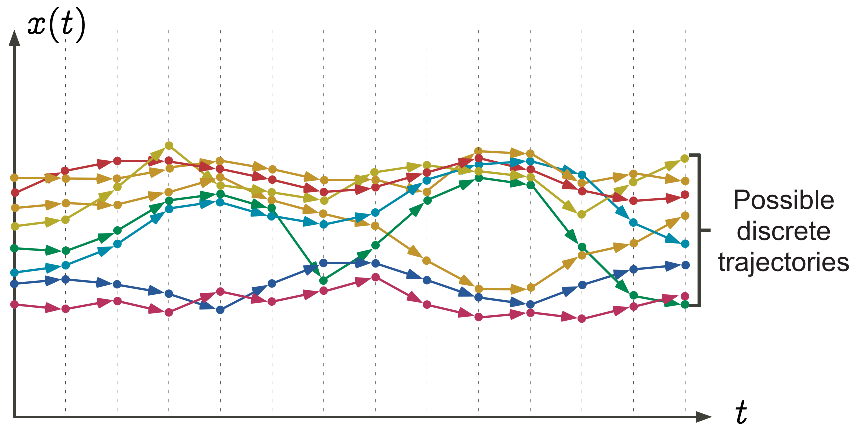
A very informal introduction to static analysis algorithms

Trace semantics

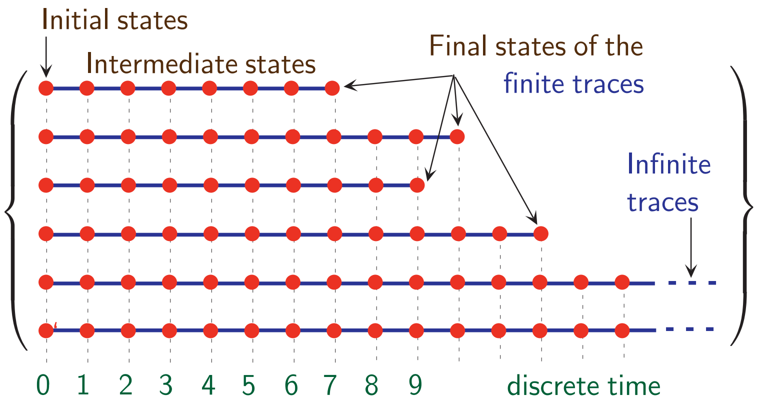
Trace semantics

- Consider (possibly infinite) traces that is series of states corresponding to executions described by discrete transitions;
- The collection of all such traces, starting from the initial states, is the trace semantics.

Graphic example: Small-steps transition semantics

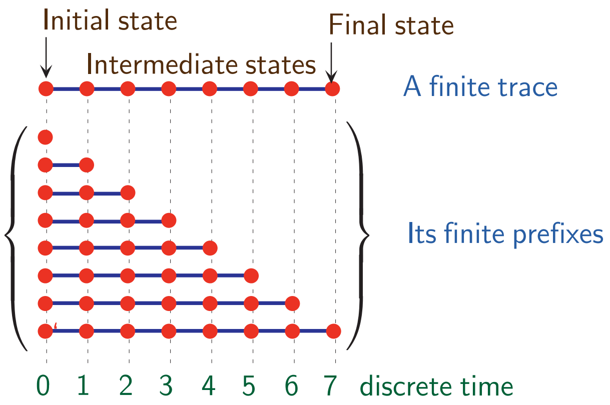


Trace semantics, intuition

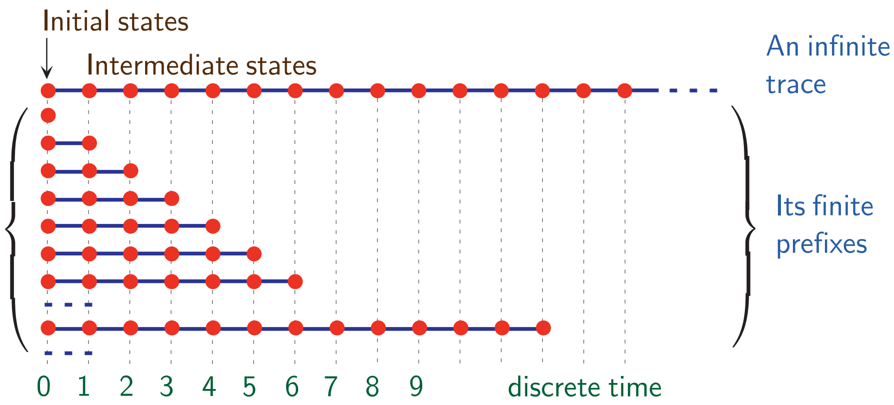


Prefix trace semantics

Prefixes of a finite trace



Prefixes of an infinite trace



Prefix trace semantics

Trace semantics: maximal finite and infinite behaviors

Prefix trace semantics: finite prefixes of the maximal behaviors

Abstraction

This is an abstraction. For example:

Trace semantics: $\{a^n b \mid n \geq 0\}$

Prefix trace semantics: $\{a^n \mid n \geq 0\} \cup \{a^n b \mid n \geq 0\}$

Is there of possible behavior with infinitely many successive a?

- Trace semantics: no
- Prefix trace semantics: I don't know

Prefix trace semantics in fixpoint form

Least Fixpoint Prefix Trace Semantics

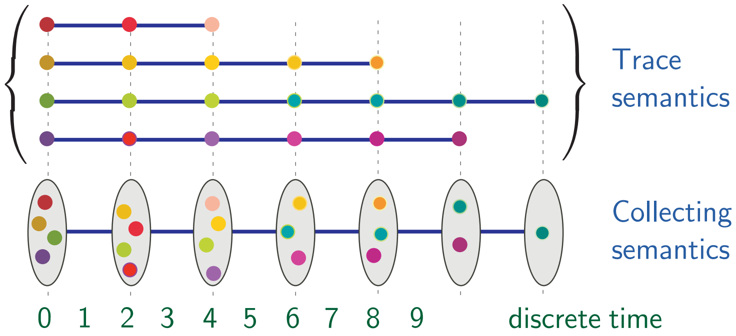
Prefixes = {• | • is an initial state}
∪ {•—...—•—• | •—...—• ∈ **Prefixes** & •—• is a transition step}

- In general, the equation **Prefixes** $= F($**Prefixes**$)$ may have multiple solutions;
- Choose the least one for subset inclusion $\subseteq$.
- Abstractions of this equation lead to effective iterative analysis algorithms.

Collecting semantics

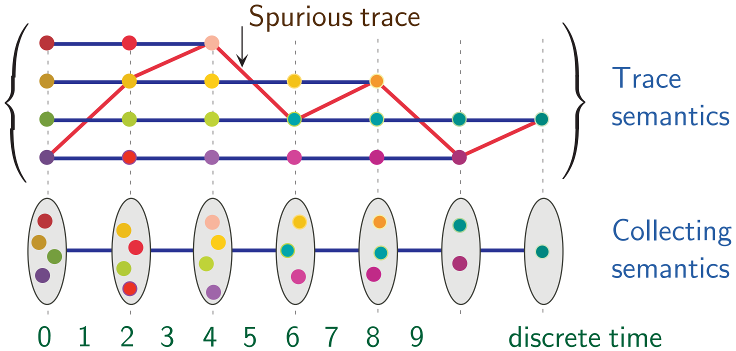
Collecting semantics

– Collect all states that can appear on some trace at any given discrete time:

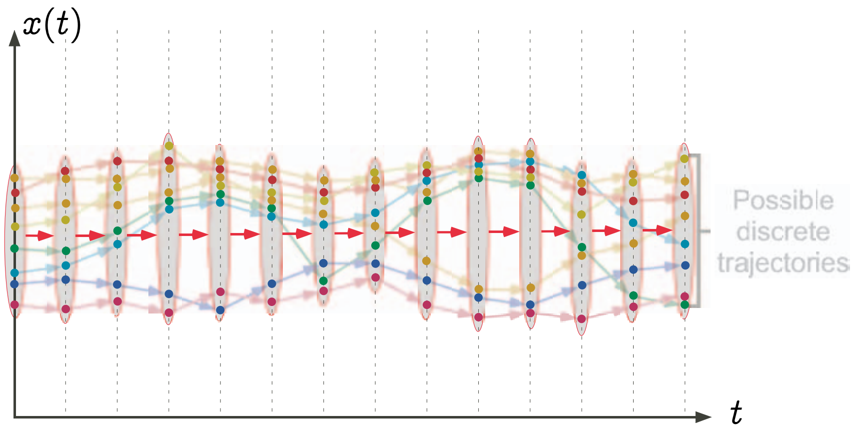


Collecting abstraction

– This is an abstraction. Does the red trace exists? Trace semantics: no, collecting semantics: I don't know.

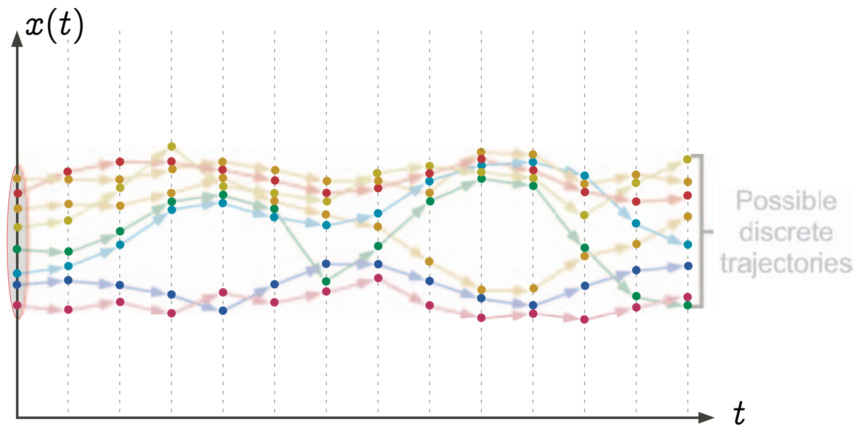


Graphic example: collecting semantics

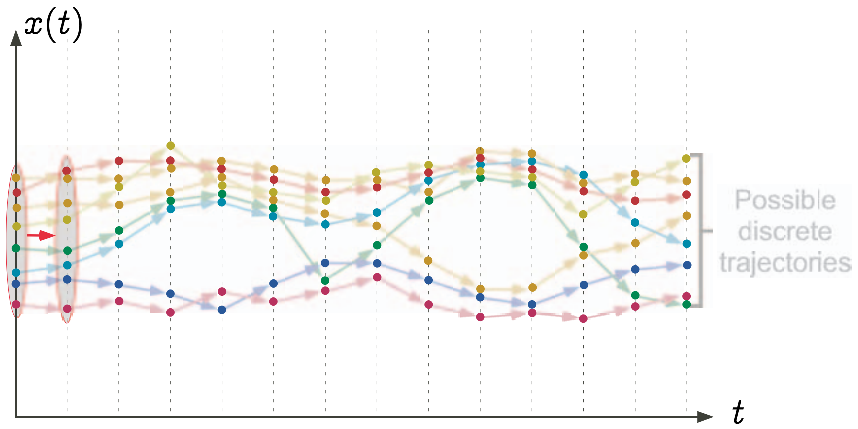


Collecting semantics in fixpoint form

Graphic example: collecting semantics in fixpoint form



Graphic example: collecting semantics in fixpoint form



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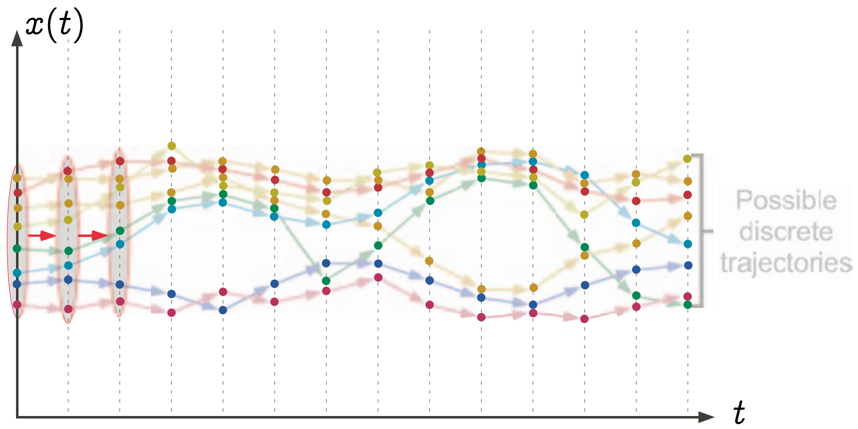

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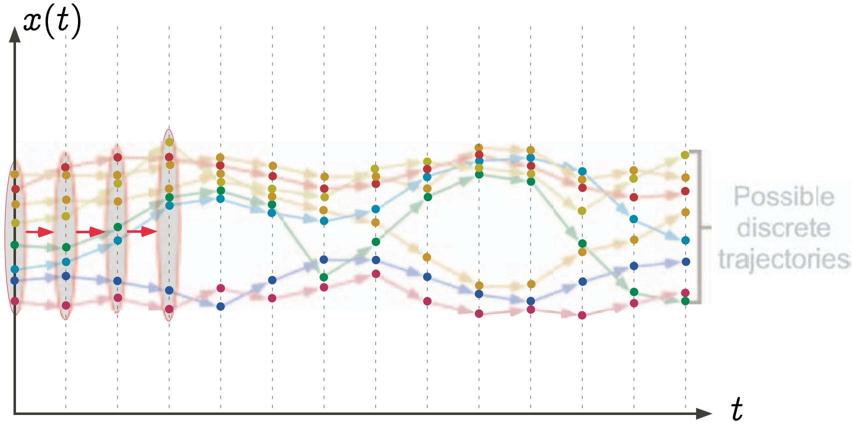

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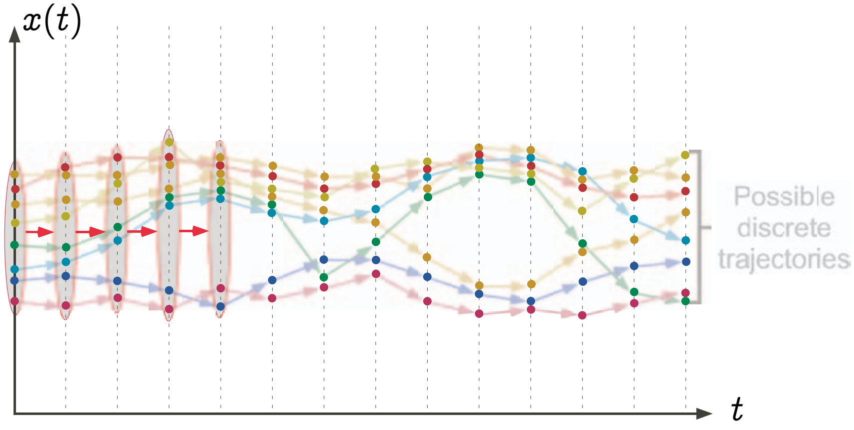

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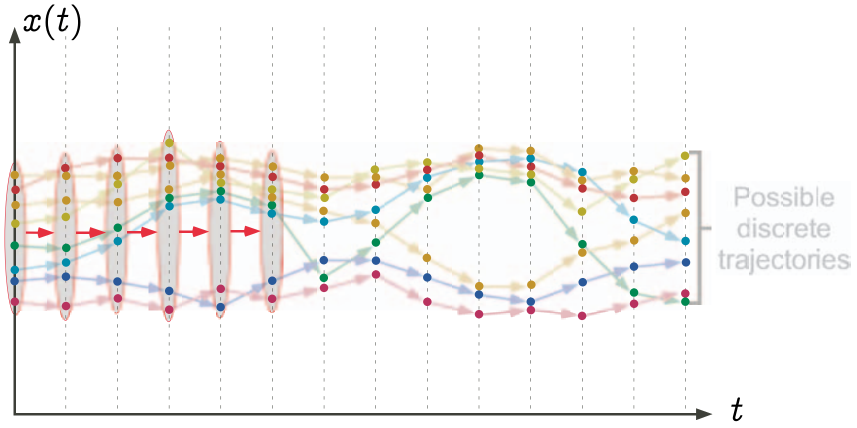

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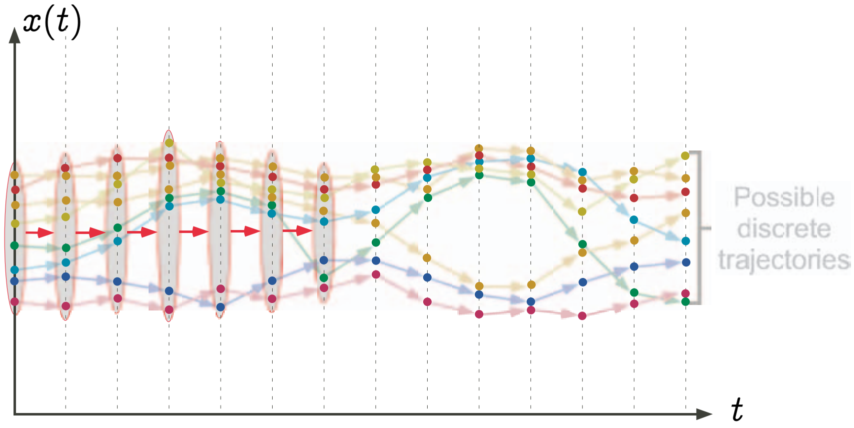
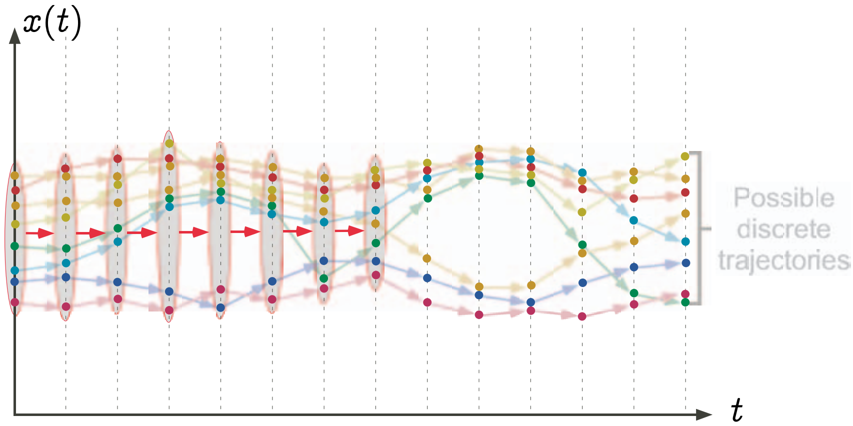


Graphic example: collecting semantics in fixpoint form



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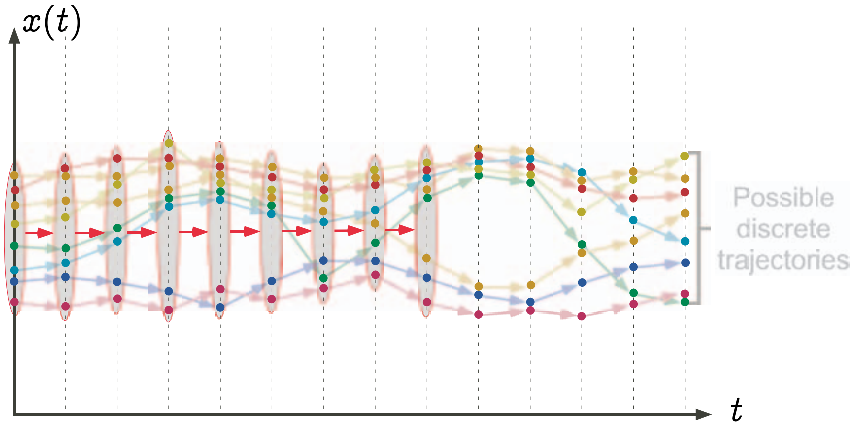

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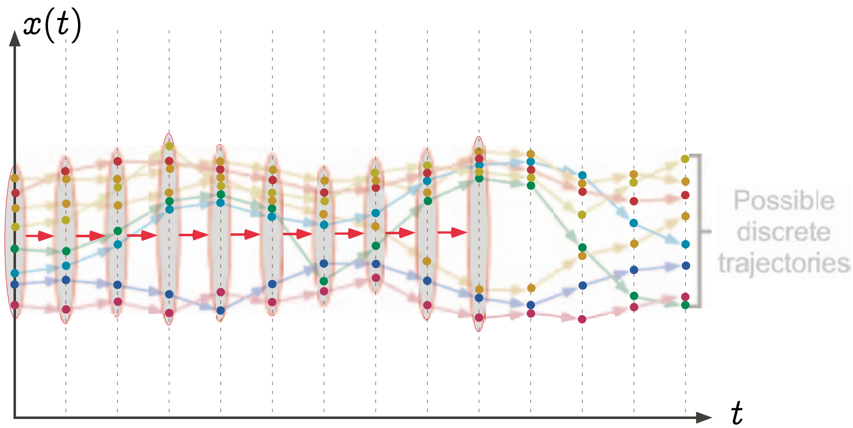
Graphic example: collecting semantics in fixpoint form



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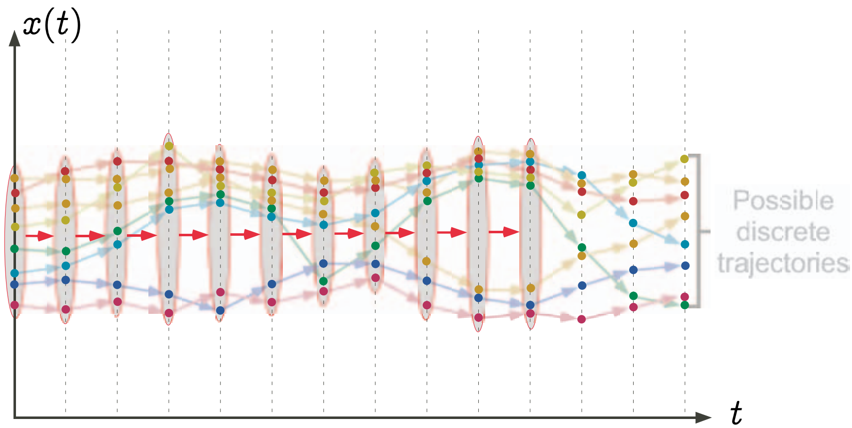
Graphic example: collecting semantics in fixpoint form



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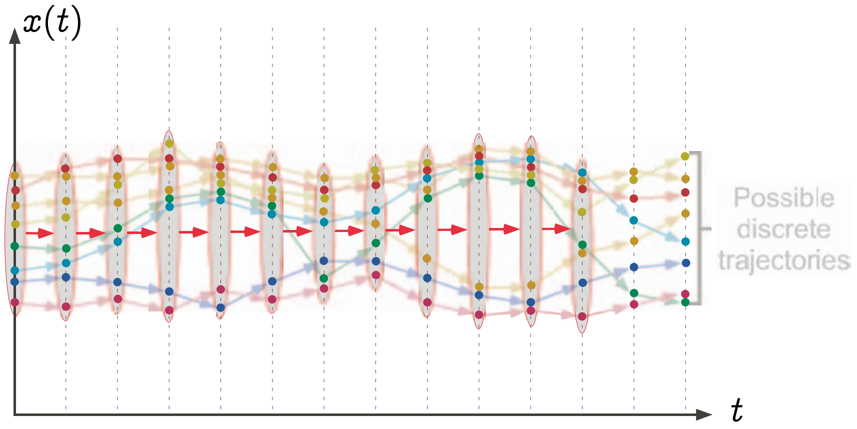

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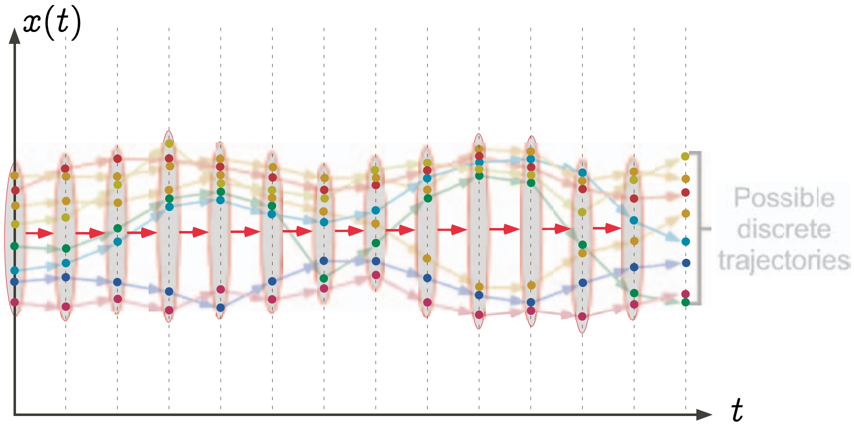

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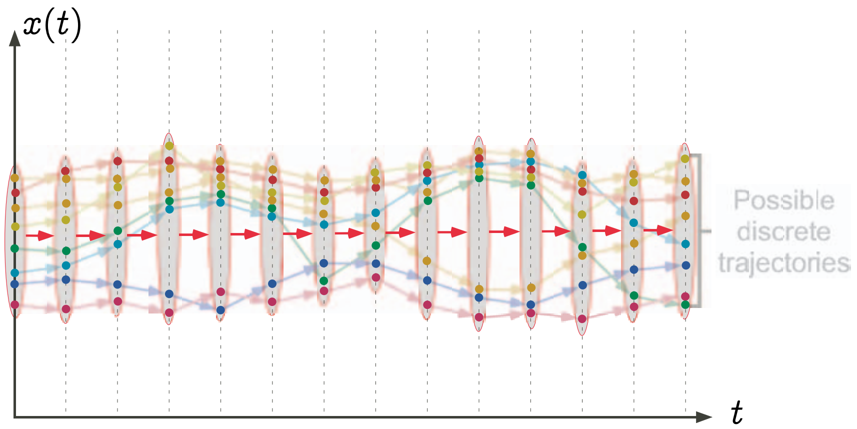
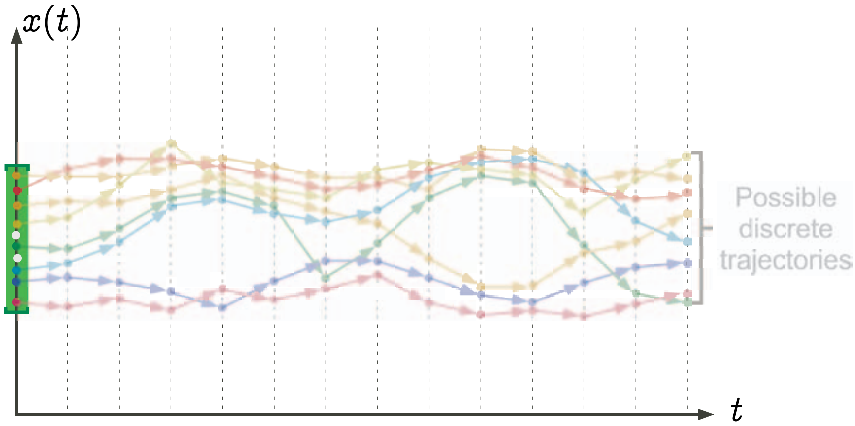


Interval Abstraction (in iterative fixpoint form)

Graphic example: traces of intervals in fixpoint form



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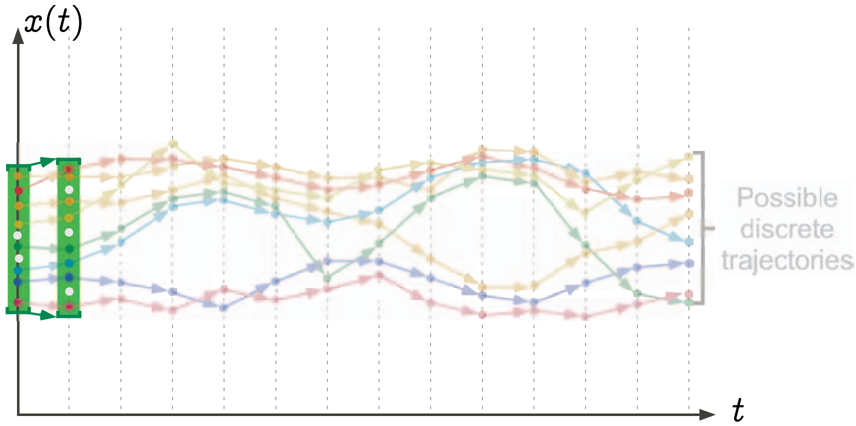

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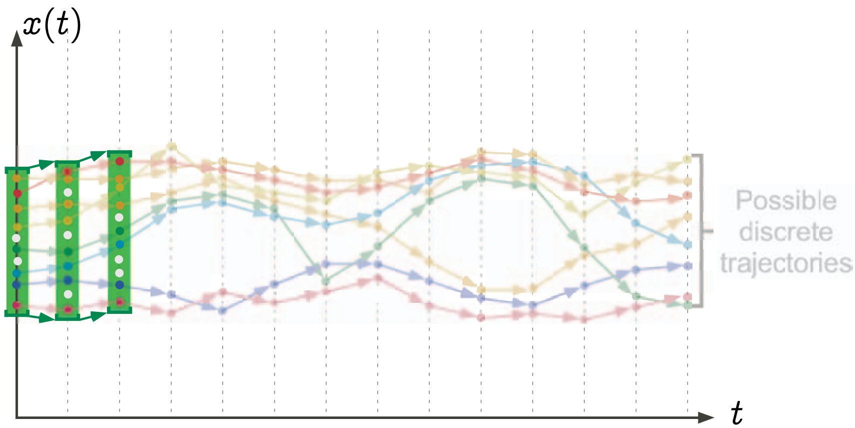

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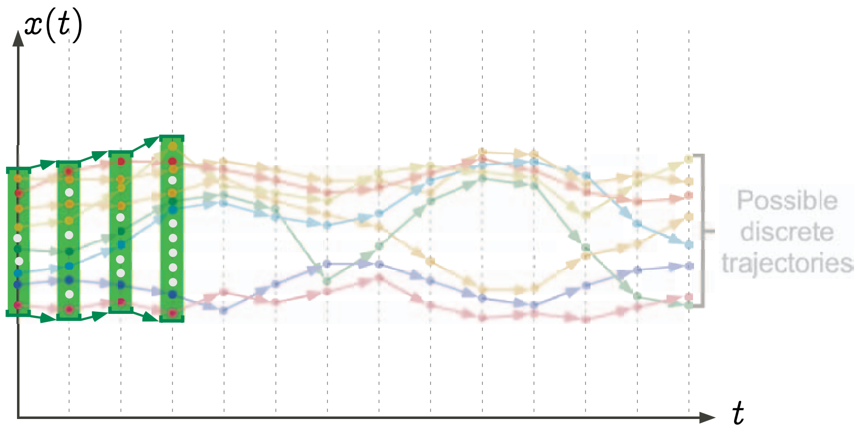

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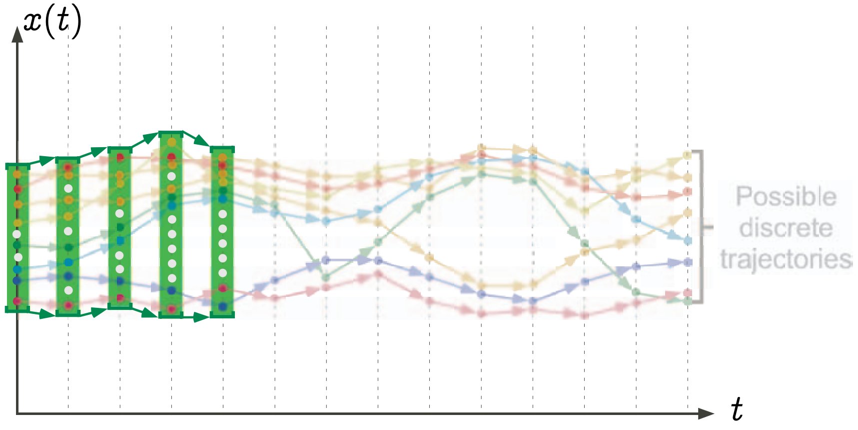

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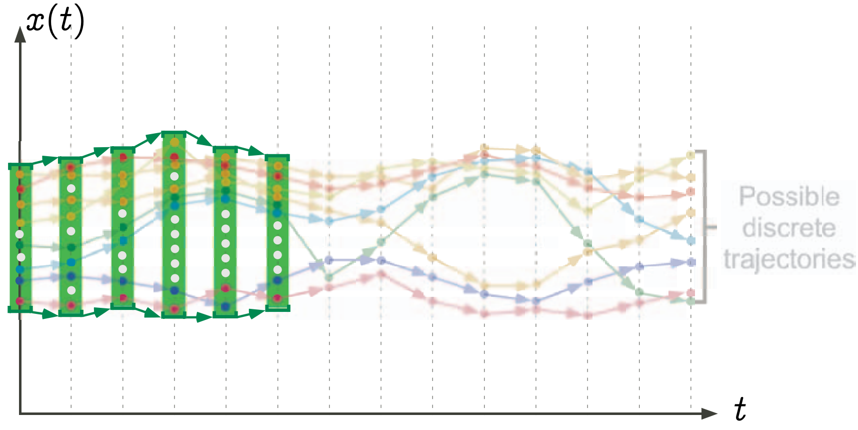

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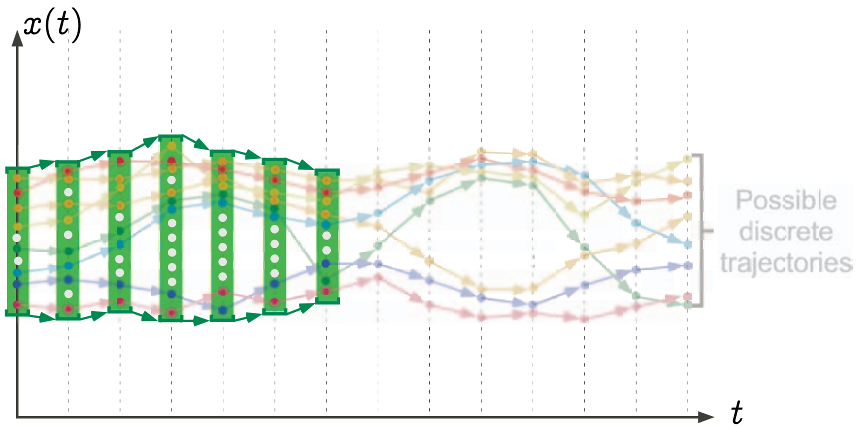
Graphic example: traces of intervals in fixpoint form



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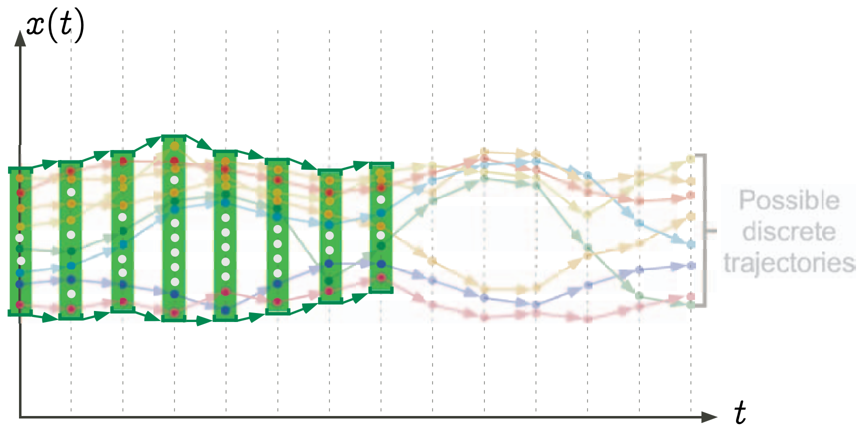

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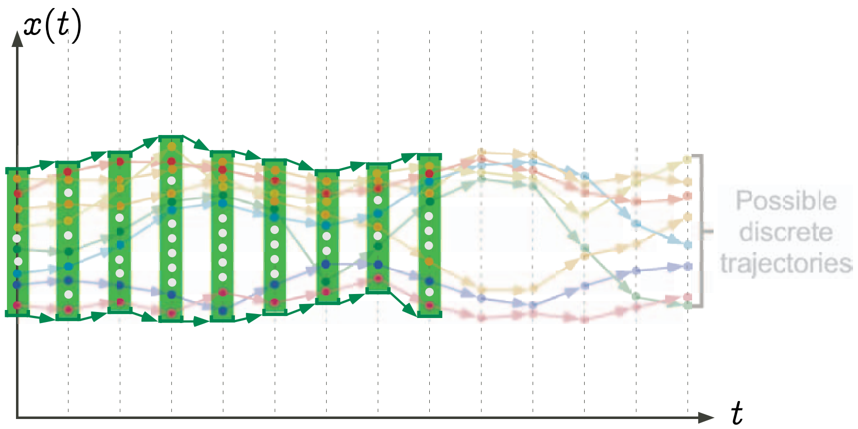

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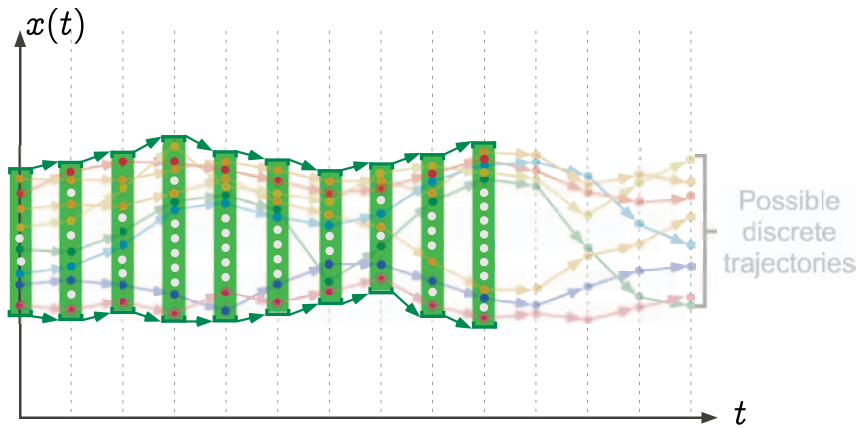

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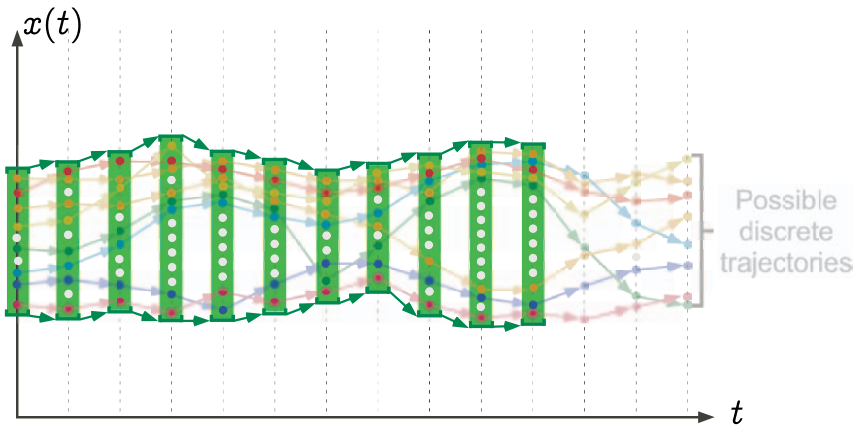

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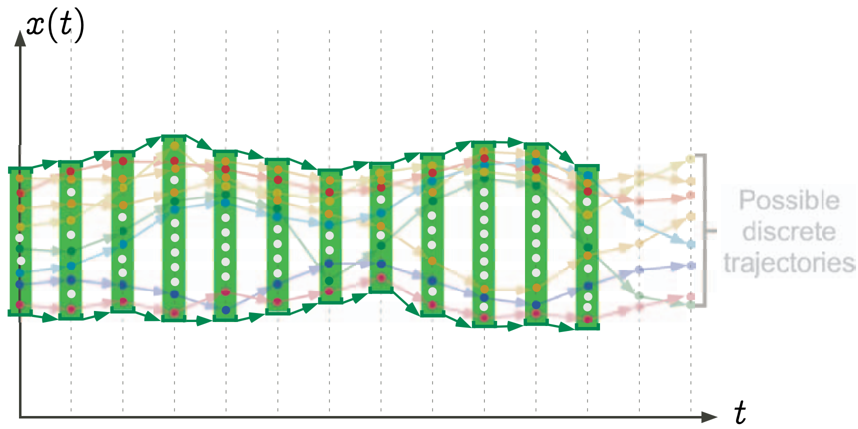

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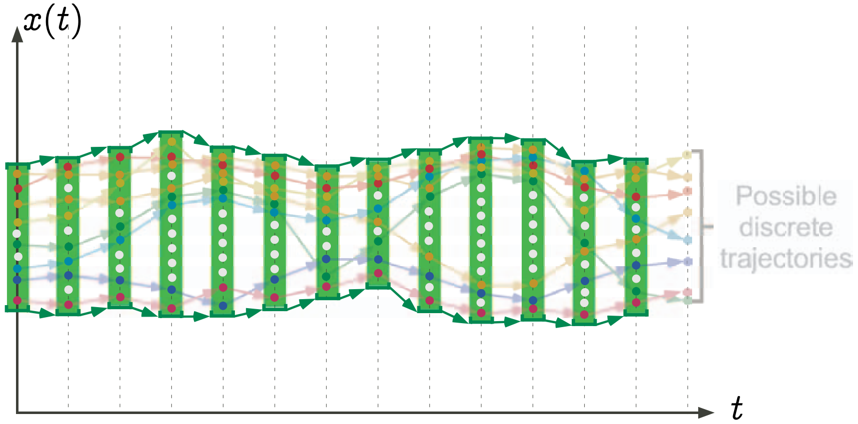

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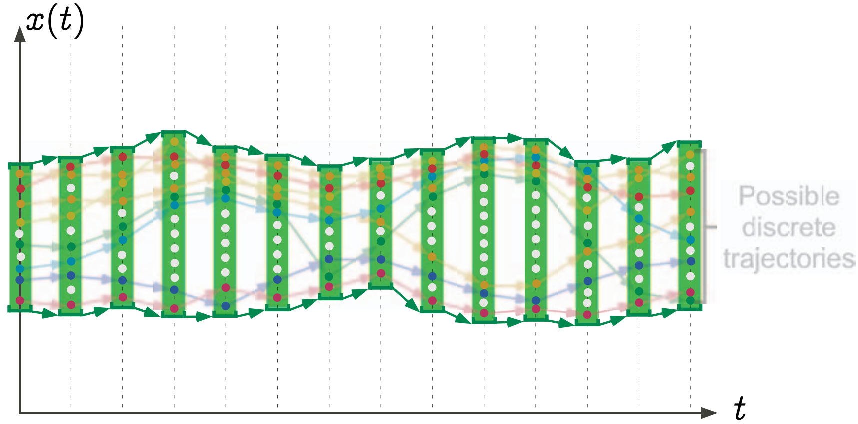

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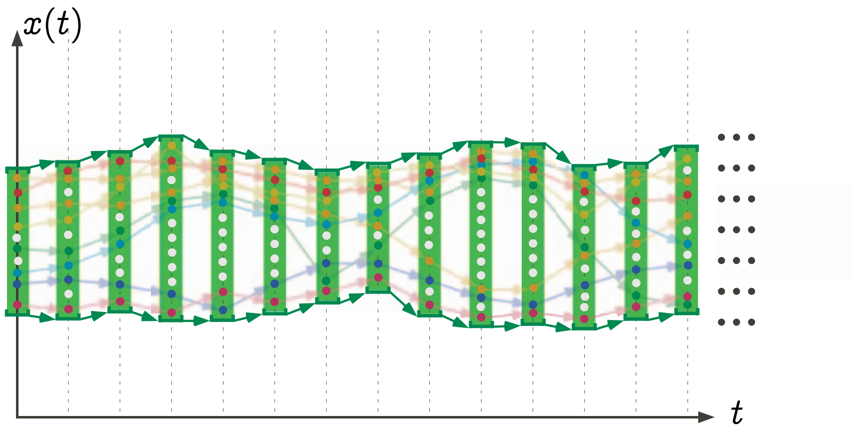
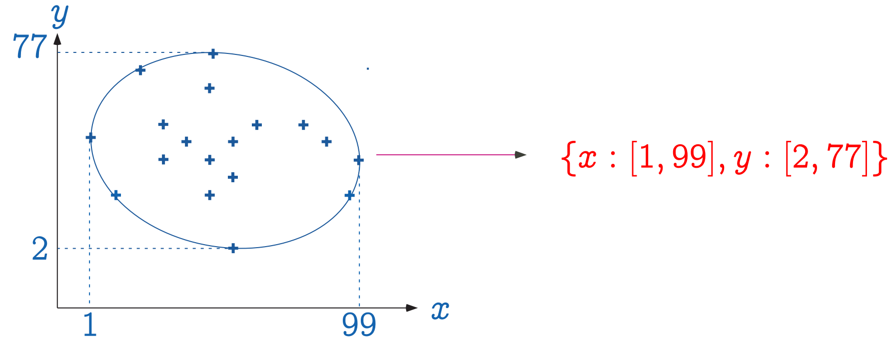


Abstraction by Galois connections

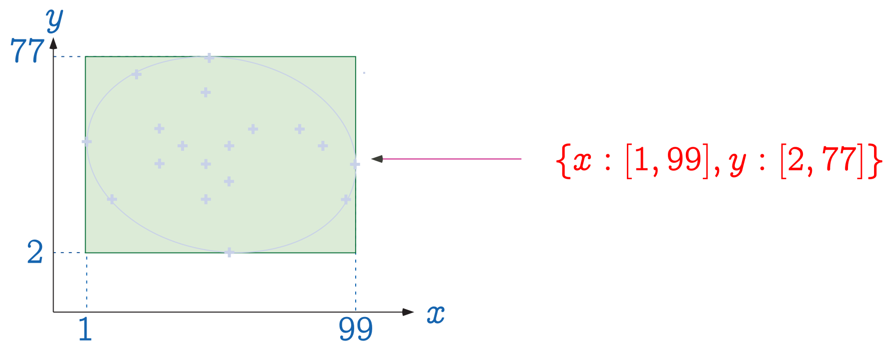
Abstracting sets (i.e. properties)

- Choose an abstract domain, replacing sets of objects (states, traces, ...) S by their abstraction $\alpha(S)$
- The abstraction function α maps a set of concrete objects to its abstract interpretation;
- The inverse concretization function γ maps an abstract set of objects to concrete ones;
- Forget no concrete objects: (abstraction from above) $S \subseteq \gamma(\alpha(S))$.

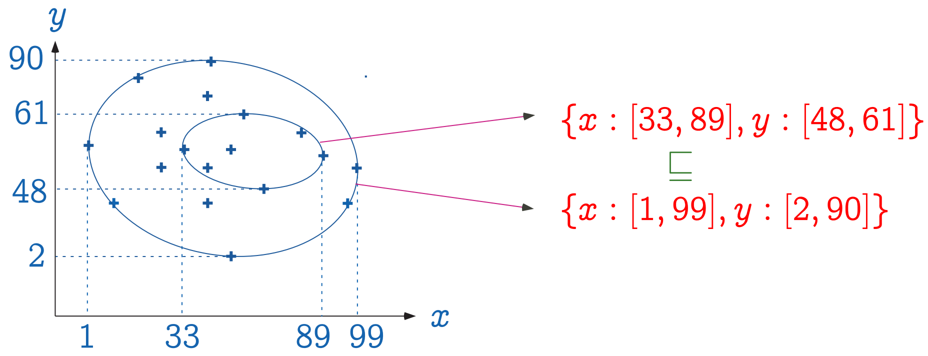
Interval abstraction α



Interval concretization γ

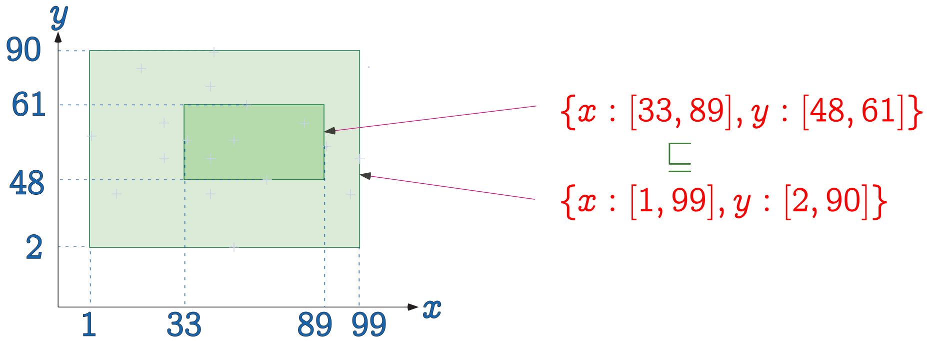


The abstraction α is monotone



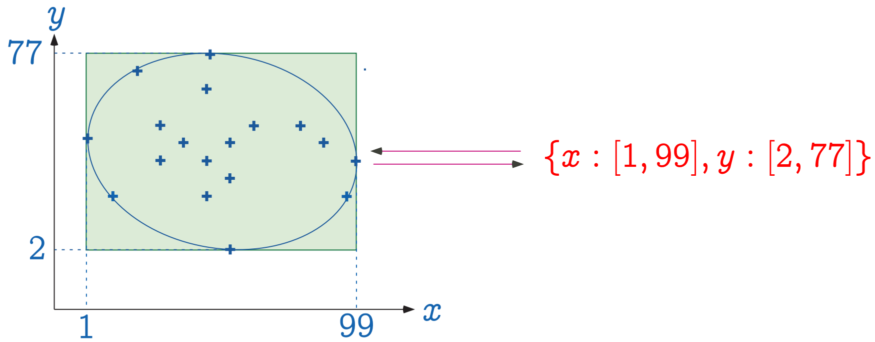
$$X \subseteq Y \Rightarrow \alpha(X) \sqsubseteq \alpha(Y)$$

The concretization γ is monotone



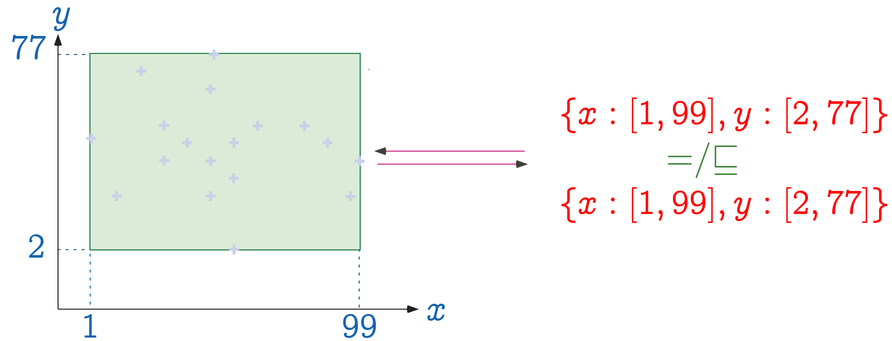
$$X \sqsubseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$$

The $\gamma \circ \alpha$ composition is extensive



$$X \subseteq \gamma \circ \alpha(X)$$

The $\alpha \circ \gamma$ composition is reductive



$$\alpha \circ \gamma(Y) = / \sqsubseteq Y$$

Correspondance between concrete and abstract properties

- The pair $\langle \alpha, \gamma \rangle$ is a Galois connection:

$$\langle \wp(S), \subseteq \rangle \xrightarrow[\alpha]{\gamma} \langle \mathcal{D}, \sqsubseteq \rangle$$

- $\langle \wp(S), \subseteq \rangle \xrightarrow[\alpha]{\gamma} \!\!\!\!\rightarrow \langle \mathcal{D}, \sqsubseteq \rangle$ when α is onto (equivalently $\alpha \circ \gamma = 1$ or γ is one-to-one).

Galois connection

$$\langle \mathcal{D}, \subseteq \rangle \xrightleftharpoons[\alpha]{\gamma} \langle \overline{\mathcal{D}}, \sqsubseteq \rangle$$

iff $\forall x, y \in \mathcal{D} : x \subseteq y \Longrightarrow \alpha(x) \sqsubseteq \alpha(y)$

$\wedge\ \forall \overline{x}, \overline{y} \in \overline{\mathcal{D}} : \overline{x} \sqsubseteq \overline{y} \Longrightarrow \gamma(\overline{x}) \subseteq \gamma(\overline{y})$

$\wedge\ \forall x \in \mathcal{D} : x \subseteq \gamma(\alpha(x))$

$\wedge\ \forall \overline{y} \in \overline{\mathcal{D}} : \alpha(\gamma(\overline{y})) \sqsubseteq \overline{x}$

iff $\forall x \in \mathcal{D}, \overline{y} \in \overline{\mathcal{D}} : \alpha(x) \sqsubseteq y \iff x \subseteq \gamma(y)$

Example: Set of traces to trace of intervals abstraction

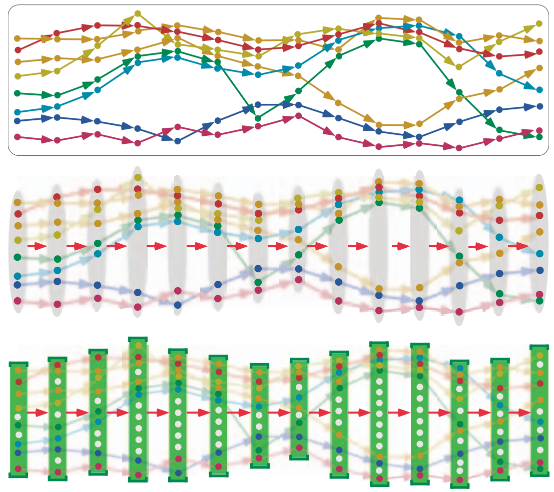
Set of traces:

$\alpha_1 \downarrow$

Trace of sets:

$\alpha_2 \downarrow$

Trace of intervals



Example: Set of traces to reachable states abstraction

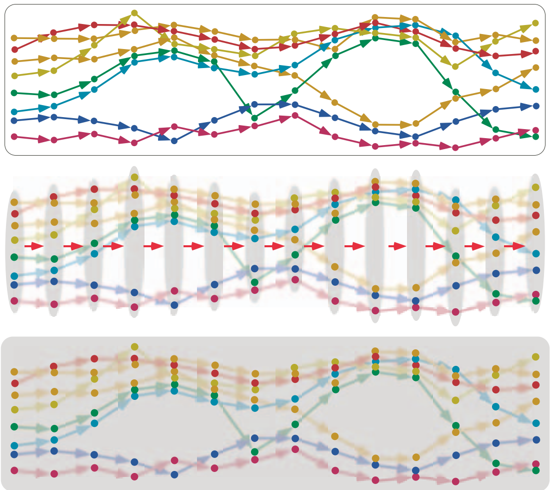
Set of traces:

$\alpha_1 \downarrow$

Trace of sets:

$\alpha_3 \downarrow$

Reachable states



Composition of Galois Connections

The composition of Galois connections:

$$\langle L, \leq \rangle \xrightleftharpoons[\alpha_1]{\gamma_1} \langle M, \sqsubseteq \rangle$$

and:

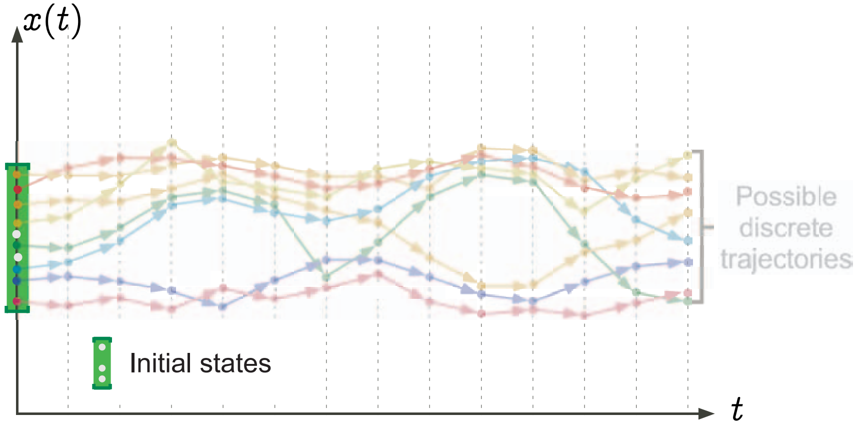
$$\langle M, \sqsubseteq \rangle \xrightleftharpoons[\alpha_2]{\gamma_2} \langle N, \preceq \rangle$$

is a Galois connection:

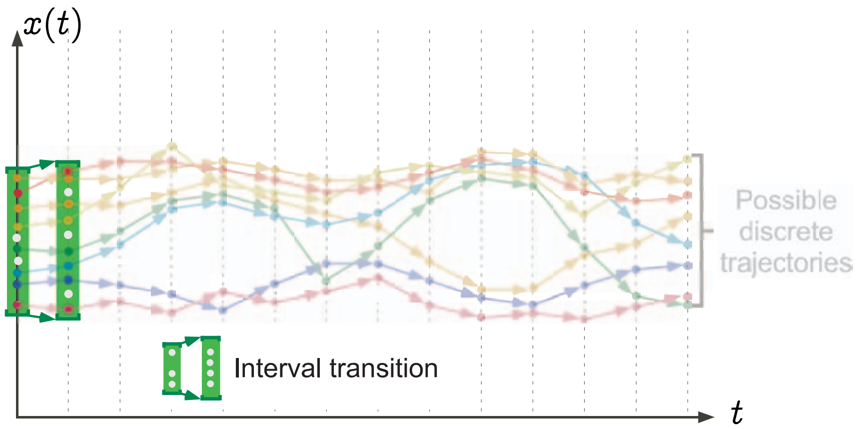
$$\langle L, \leq \rangle \xrightleftharpoons[\alpha_2 \circ \alpha_1]{\gamma_1 \circ \gamma_2} \langle N, \preceq \rangle$$

Convergence acceleration by widening/narrowing

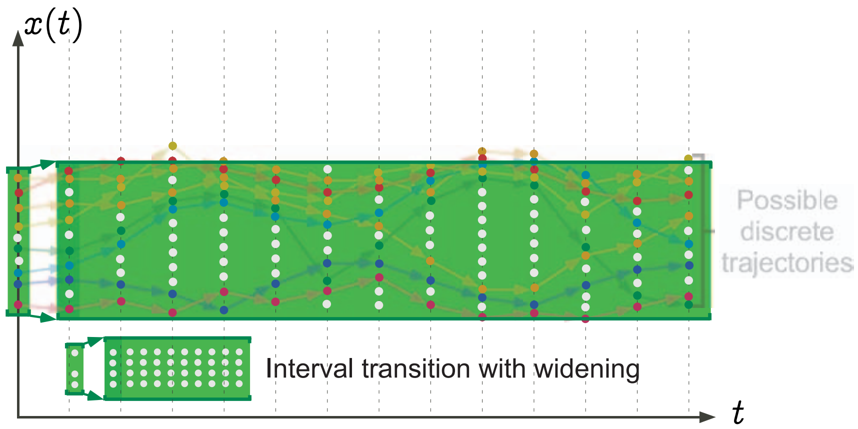
Graphic example: upward iteration with widening



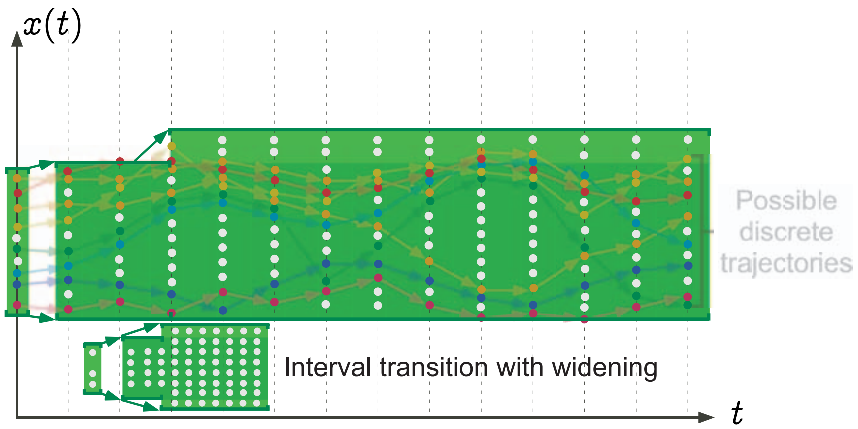
Graphic example: upward iteration with widening



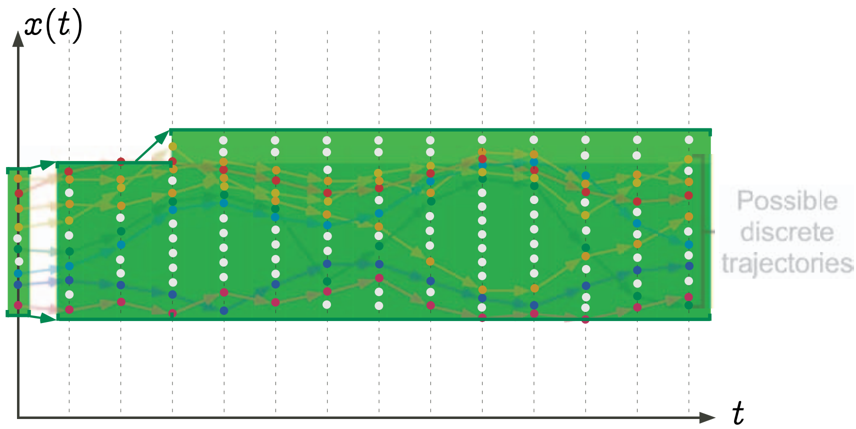
Graphic example: upward iteration with widening



Graphic example: upward iteration with widening



Graphic example: stability of the upward iteration



Interval widening

- $\overline{L} = \{\bot\}\cup\{[\ell, u] \mid \ell, u \in \mathbb{Z}\cup\{-\infty\}\wedge u \in \mathbb{Z}\cup\{\}\wedge \ell \leq u\}$
- The widening extrapolates unstable bounds to infinity:

$$\bot \mathbin{\nabla} X = X$$

$$X \mathbin{\nabla} \bot = X$$

$$[\ell_0,\ u_0] \mathbin{\nabla} [\ell_1,\ u_1] = [\text{if } \ell_1 < \ell_0 \text{ then } -\infty \text{ else } \ell_0,\ \text{if } u_1 > u_0 \text{ then } +\infty \text{ else } u_0]$$

Not monotone. For example $[0,\ 1] \sqsubseteq [0,\ 2]$ but $[0,\ 1] \mathbin{\nabla} [0,\ 2] = [0,\ +\infty] \not\sqsubseteq [0,\ 2] = [0,\ 2] \mathbin{\nabla} [0,\ 2]$

Example: Interval analysis (1975)

Program to be analyzed:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

Example: Interval analysis (1975)

Equations (abstract interpretation of the semantics):

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

Example: Interval analysis (1975)

Resolution by chaotic increasing iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = \emptyset \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = \emptyset \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,1] \\ X_3 = \emptyset \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,1] \\ X_3 = [2,2] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,2] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,2] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,3] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,3] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,4] \\ X_3 = [2,4] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,4] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,5] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Increasing chaotic iteration: **convergence !!!!!!!**

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,5] \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Convergence speed-up by widening:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,+\infty] \quad \Leftarrow \text{widening} \\ X_3 = [2,6] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,+\infty] \\ X_3 = [2,+\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1,9999] \\ X_3 = [2,+\infty] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Decreasing chaotic iteration:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = \emptyset \end{cases}$$

Example: Interval analysis (1975)

Final solution:

```
   x := 1;
1:
   while x < 10000 do
2:
       x := x + 1
3:
   od;
4:
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

Example: Interval analysis (1975)

Result of the interval analysis:

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}
       x := x + 1
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

$$\begin{cases} X_1 = [1,1] \\ X_2 = (X_1 \cup X_3) \cap [-\infty, 9999] \\ X_3 = X_2 \oplus [1,1] \\ X_4 = (X_1 \cup X_3) \cap [10000, +\infty] \end{cases}$$

$$\begin{cases} X_1 = [1,1] \\ X_2 = [1, 9999] \\ X_3 = [2, +10000] \\ X_4 = [+10000, +10000] \end{cases}$$

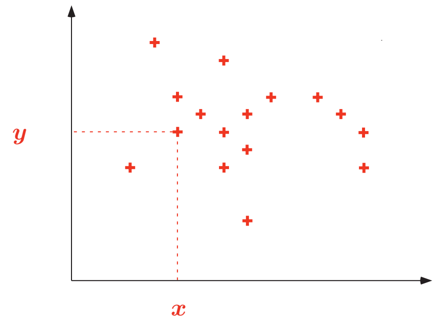
Example: Interval analysis (1975)

Checking absence of runtime errors with interval analysis:

```
   x := 1;
1: {x = 1}
   while x < 10000 do
2: {x ∈ [1, 9999]}
       x := x + 1          ⟵ no overflow
3: {x ∈ [2, +10000]}
   od;
4: {x = 10000}
```

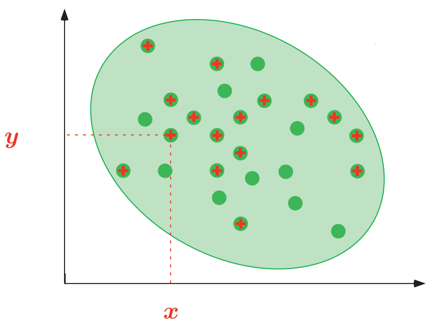
Refinement of abstractions

Approximations of an [in]finite set of points:



$\{\ldots, \langle 19,\ 77\rangle, \ldots,$
$\langle 20,\ 03\rangle, \ldots\}$

Approximations of an [in]finite set of points: from above



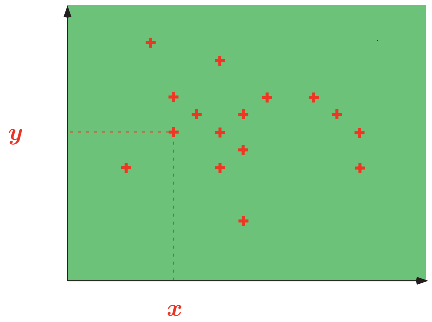
$\{\ldots, \langle 19,\ 77\rangle, \ldots,$

$\langle 20,\ 03\rangle, \langle ?,\ ?\rangle, \ldots\}$

From Below: dual[3] + combinations.

[3] Trivial for finite states (liveness model-checking), more difficult for infinite states (variant functions).

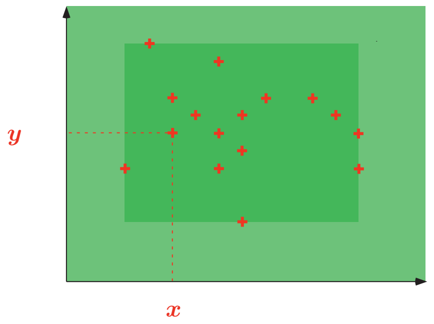
Effective computable approximations of an [in]finite set of points; Signs[4]



$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

[4] P. Cousot & R. Cousot. *Systematic design of program analysis frameworks.* ACM POPL'79, pp. 269–282, 1979.

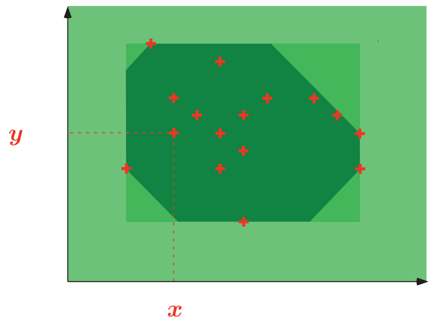
Effective computable approximations of an [in]finite set of points; Intervals [5]



$$\begin{cases} x \in [19,\ 77] \\ y \in [20,\ 03] \end{cases}$$

[5] P. Cousot & R. Cousot. *Static determination of dynamic properties of programs.* Proc. 2nd Int. Symp. on Programming, Dunod, 1976.

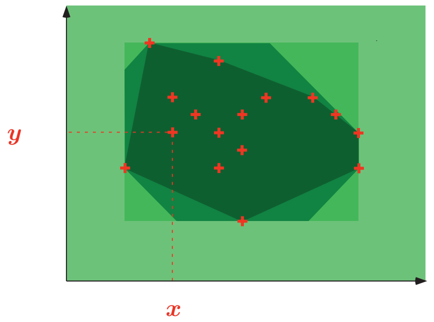
Effective computable approximations of an [in]finite set of points; Octagons [6]



$$\begin{cases} 1 \leq x \leq 9 \\ x + y \leq 77 \\ 1 \leq y \leq 9 \\ x - y \leq 99 \end{cases}$$

[6] A. Miné. *A New Numerical Abstract Domain Based on Difference-Bound Matrices.* PADO'2001. LNCS 2053, pp. 155–172. Springer 2001. See the *The Octagon Abstract Domain Library* on `http://www.di.ens.fr/~mine/oct/`

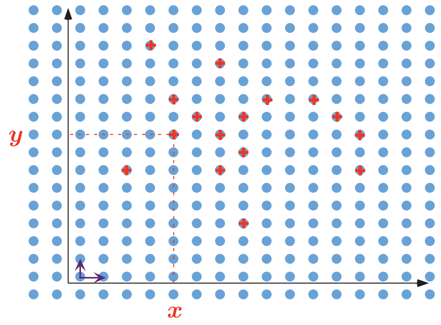
Effective computable approximations of an [in]finite set of points; Polyhedra [7]



$$\begin{cases} 19x + 77y \leq 2004 \\ 20x + 03y \geq 0 \end{cases}$$

[7] P. Cousot & N. Halbwachs. *Automatic discovery of linear restraints among variables of a program.* ACM POPL, 1978, pp. 84–97.

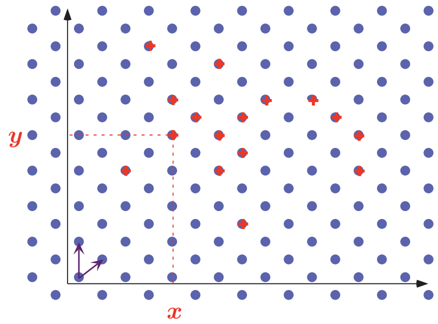
Effective computable approximations of an [in]finite set of points; Simple congruences [8]



$$\begin{cases} x = 19 \bmod 77 \\ y = 20 \bmod 99 \end{cases}$$

[8] Ph. Granger. *Static Analysis of Arithmetical Congruences.* Int. J. Comput. Math. 30, 1989, pp. 165–190.

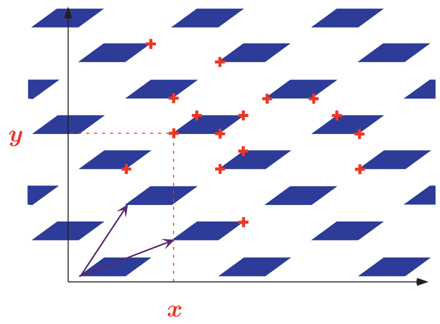
Effective computable approximations of an [in]finite set of points; Linear congruences [9]



$$\begin{cases} 1x + 9y = 7 \bmod 8 \\ 2x - 1y = 9 \bmod 9 \end{cases}$$

[9] Ph. Granger. *Static Analysis of Linear Congruence Equalities among Variables of a Program.* TAPSOFT '91, pp. 169–192. LNCS 493, Springer, 1991.

Effective computable approximations of an [in]finite set of points; Trapezoidal linear congruences [10]

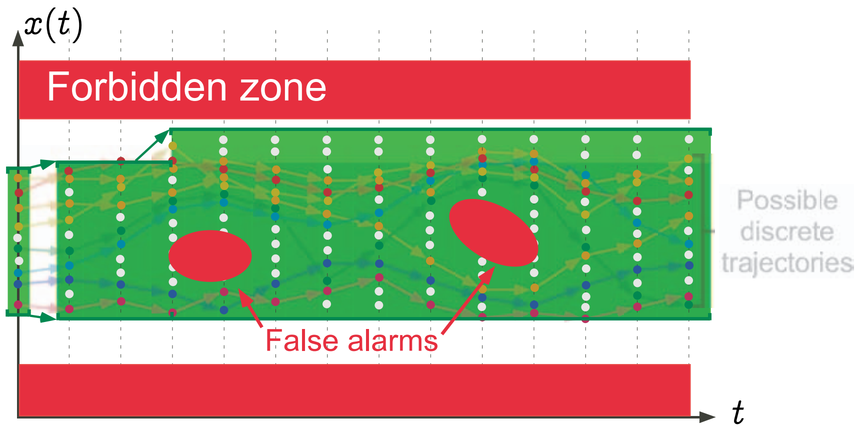


$$\begin{cases} 1x + 9y \in [0, 77] \bmod 10 \\ 2x - 1y \in [0, 99] \bmod 11 \end{cases}$$

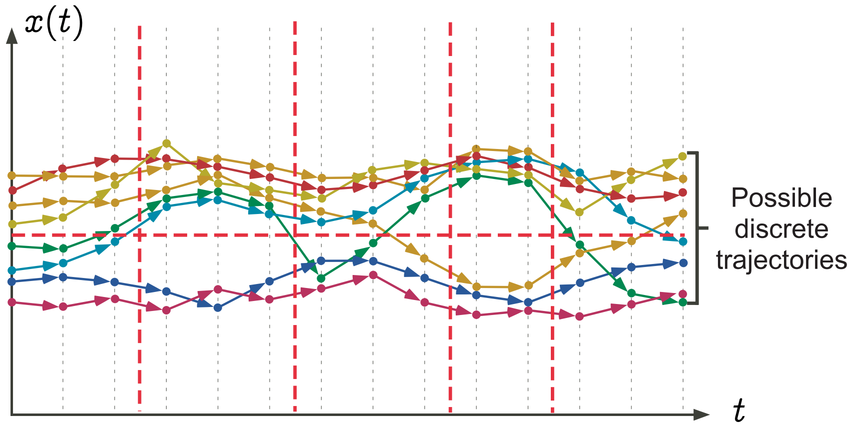
[10] F. Masdupuy. *Array Operations Abstraction Using Semantic Analysis of Trapezoid Congruences.* ACM ICS '92.

Refinement of iterates

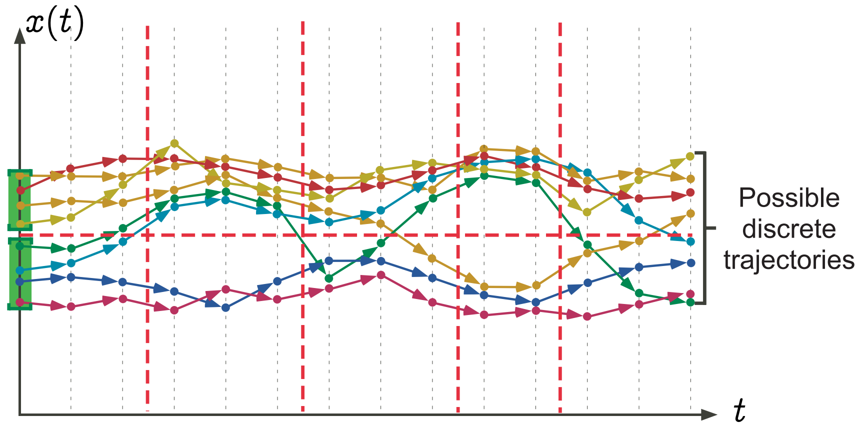
Graphic example: Refinement required by false alarms



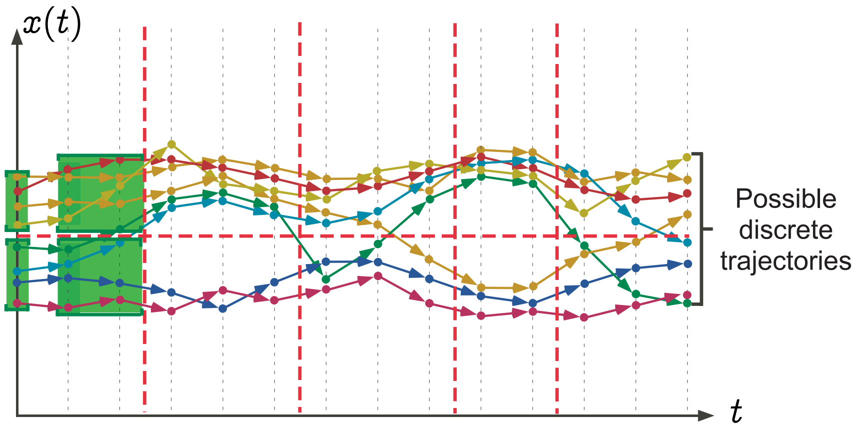
Graphic example: Partitionning



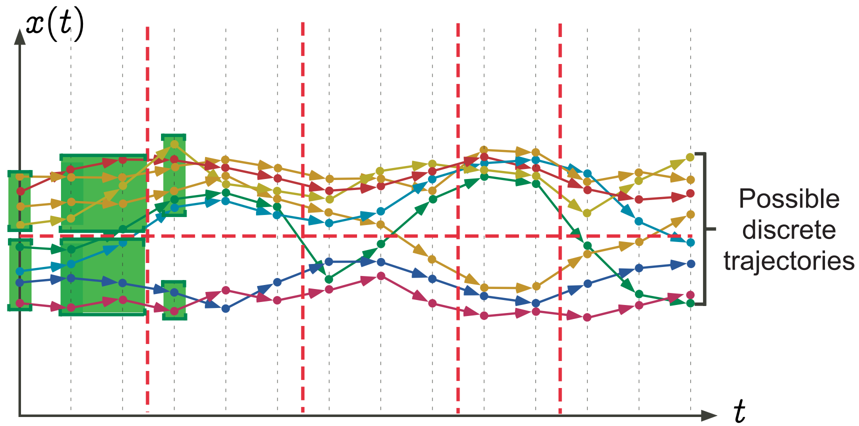
Graphic example: partitionned upward iteration with widening



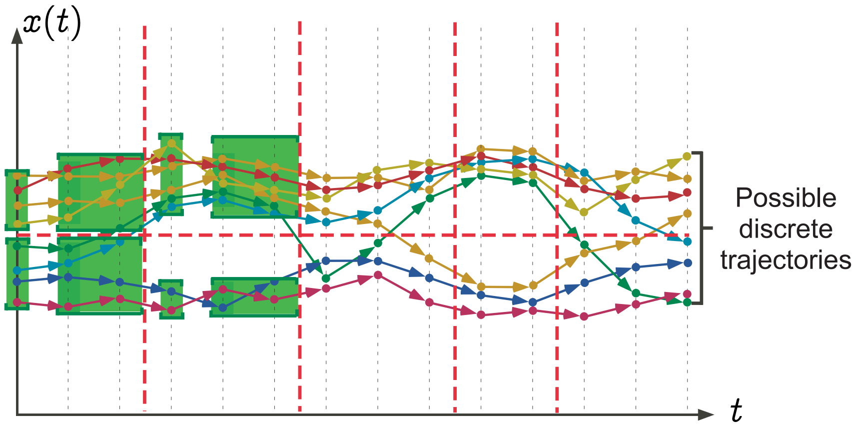
Graphic example: partitionned upward iteration with widening



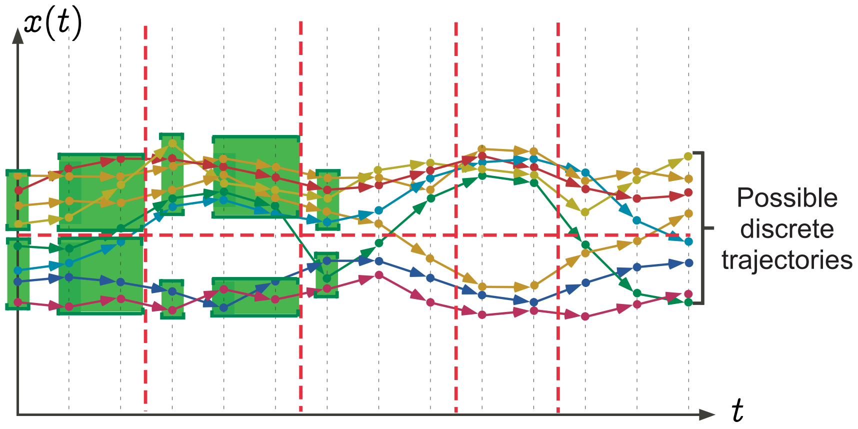
Graphic example: partitionned upward iteration with widening



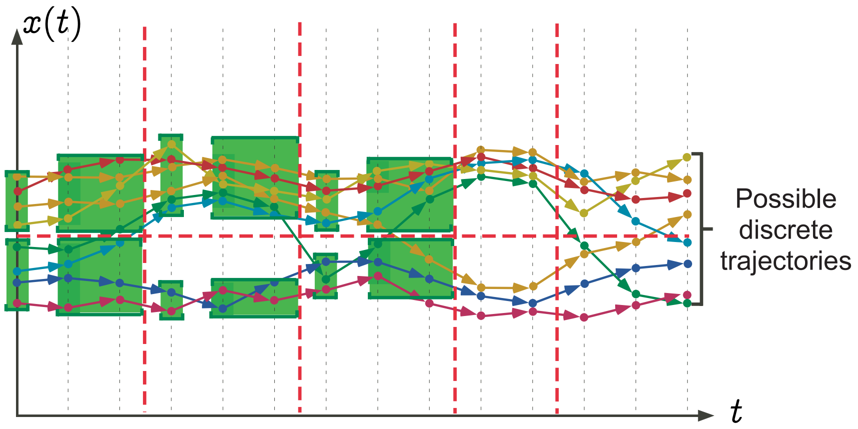
Graphic example: partitionned upward iteration with widening



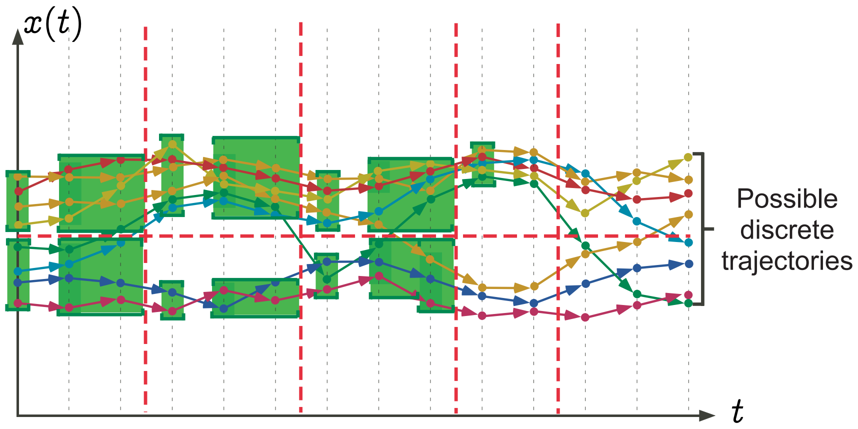
Graphic example: partitionned upward iteration with widening



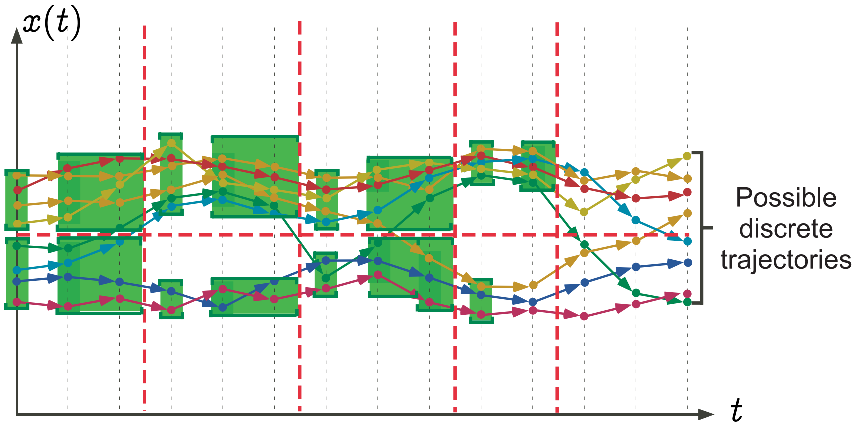
Graphic example: partitionned upward iteration with widening



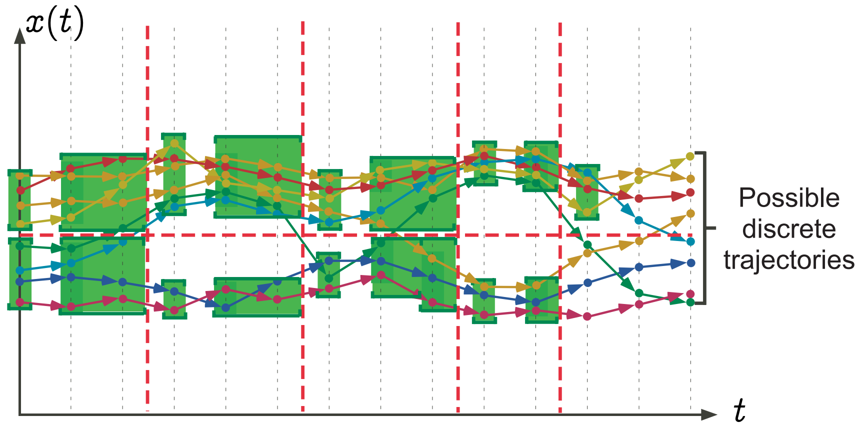
Graphic example: partitionned upward iteration with widening



Graphic example: partitionned upward iteration with widening



Graphic example: partitionned upward iteration with widening



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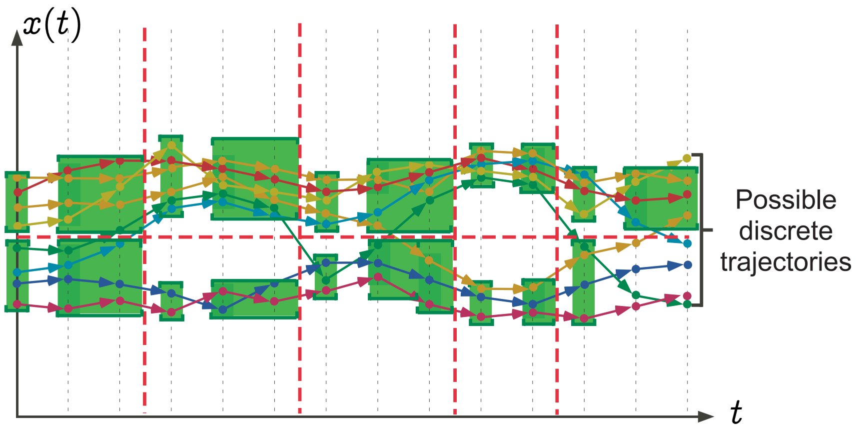
Graphic example: partitionned upward iteration with widening



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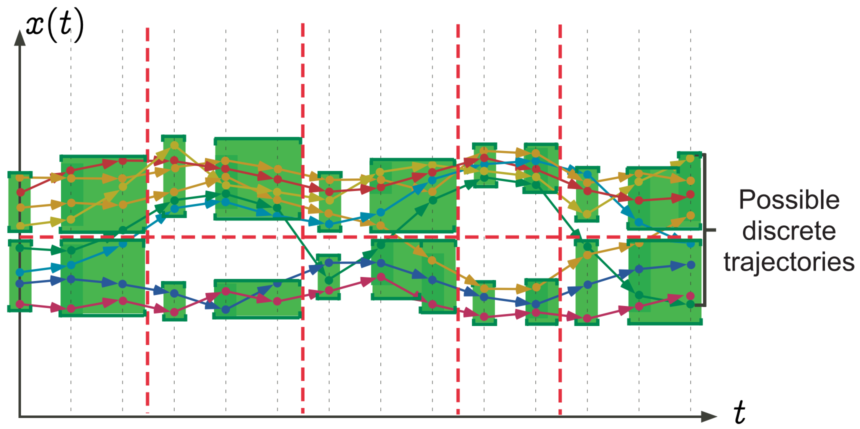
Graphic example: partitionned upward iteration with widening



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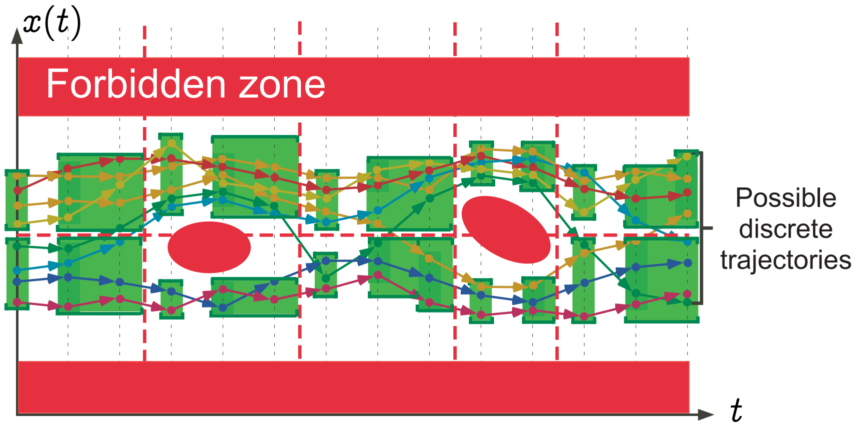
Graphic example: safety verification

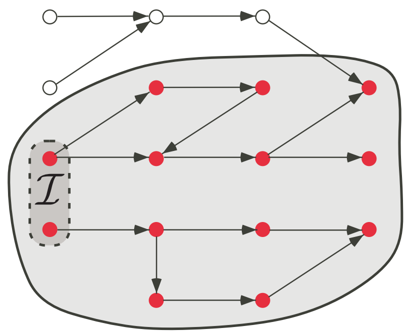


Interval widening with threshold set

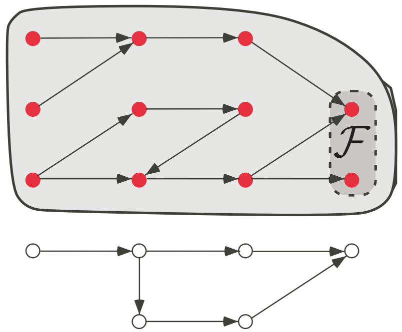
- The threshold set T is a finite set of numbers (plus $+\infty$ and $-\infty$),
- $$[a, b] \mathbin{\nabla_T} [a', b'] = [\mathit{if}\ a' < a\ \mathit{then}\ \max\{\ell \in T \mid \ell \leq a'\}\ \mathit{else}\ a,\ \mathit{if}\ b' > b\ \mathit{then}\ \min\{h \in T \mid h \geq b'\}\ \mathit{else}\ b]\ .$$
- Examples (intervals):
  - sign analysis: $T = \{-\infty, 0, +\infty\}$;
  - strict sign analysis: $T = \{-\infty, -1, 0, +1, +\infty\}$;
- T is a parameter of the analysis.

Combinations of abstractions

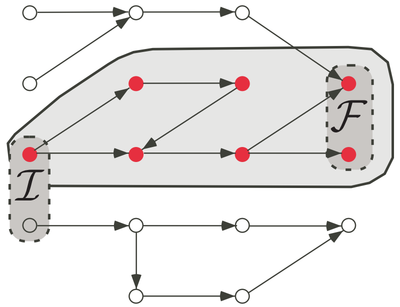
Forward/reachability analysis



Backward/ancestry analysis



Iterated forward/backward analysis



Example of iterated forward/backward analysis

Arithmetical mean of two integers x and y:

```
{x>=y}
  while (x <> y) do
    {x>=y+2}
      x := x - 1;
    {x>=y+1}
      y := y + 1
    {x>=y}
  od
{x=y}
```

Necessarily $x \geq y$ for proper termination

Example of iterated forward/backward analysis

Adding an auxiliary counter k decremented in the loop body and asserted to be null on loop exit:

```
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

Moreover the difference of x and y must be even for proper termination

Applications of abstract interpretation

Theoretical applications of abstract interpretation

– **Static Program Analysis** [POPL '77,78,79] inluding **Data-flow Analysis** [POPL '79,00], **Set-based Analysis** [FPCA '95], etc
– **Syntax Analysis** [TCS 290(1) 2002]
– **Hierarchies of Semantics (including Proofs)** [POPL '92, TCS 277(1–2) 2002]
– **Typing** [POPL '97]
– **Model Checking** [POPL '00]
– **Program Transformation** [POPL '02]
– **Software watermarking** [POPL '04]

Industrial applications of abstract interpretation

– **Program analysis and manipulation**: a small rate of false alarms is acceptable
  - AiT: worst case execution time [11]
  - StackAnalyzer: stack usage analysis [11]
– **Program verification**: no false alarms is acceptable
  - TVLA: A system for generating abstract interpreters
  - Astrée: verification of absence of run-time errors [11]

[11] applied to the primary flight control software of the Airbus A340/600 and A380 fly-by-wire systems

Bibliography

Seminal papers

– Patrick Cousot & Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In 4th Symp. on Principles of Programming Languages, pages 238—252. ACM Press, 1977.

– Patrick Cousot & Nicolas Halbwachs. Automatic discovery of linear restraints among variables of a program. In 5th Symp. on Principles of Programming Languages, pages 84—97. ACM Press, 1978.

– Patrick Cousot & Radhia Cousot. Systematic design of program analysis frameworks. In 6th Symp. on Principles of Programming Languages pages 269—282. ACM Press, 1979.

Recent surveys

– Patrick Cousot. Interprétation abstraite. Technique et Science Informatique, Vol. 19, Nb 1-2-3. Janvier 2000, Hermès, Paris, France. pp. 155-164.

– Patrick Cousot. Abstract Interpretation Based Formal Methods and Future Challenges. In Informatics, 10 Years Back — 10 Years Ahead, R. Wilhelm (Ed.), LNCS 2000, pp. 138-156, 2001.

– Patrick Cousot & Radhia Cousot. Abstract Interpretation Based Verification of Embedded Software: Problems and Perspectives. In Proc. 1st Int. Workshop on Embedded Software, EMSOFT 2001, T.A. Henzinger & C.M. Kirsch (Eds.), LNCS 2211, pp. 97–113. Springer, 2001.

Course Content

Anticipated Content of Course 16.399: Abstract Interpretation

– Today : an informal overview of abstract interpretation;

– The software verification problem (undecidability, complexity, test, simulation, specification, formal methods (deductive methods, model-checking, static analysis) and their limitations, intuitive notion of approximation, false alarms);

– Mathematical foundations (naive set theory, first order classical logic, lattice theory, fixpoints);

– Semantics of programming languages (abstract syntax, operational semantics, inductive definitions, example of a simple imperative language, grammar and interpretor of the language, trace semantics);
– Program specification and manual proofs (safety properties, Hoare logic, predicate transformers, liveness properties, linear-time temporal logic (LTL));
– Order-theoretic approximation (abstraction, closures, Galois connections, fixpoint abstraction, widening, narrowing, reduced product, absence of best approximation, refinement);

– Principle of static analysis by abstract interpretation (reachability analysis of a transition system, finite approximation, model-checking, infinite approximation, static analysis, program-based versus language-based analysis, limitations of finite approximations);
– Design of a generic structural abstract interpreter (collecting semantics, non-relational and relational analysis, convergence acceleration by wideing/narrowing);
– Static analysis (forward reachability analysis, backward analysis, iterated forward/backward analysis, inevitability analysis, termination)

– Numerical abstract domains (intervals, affine equalities, congruences, octagons, polyhedra);
– Symbolic abstract domains (abstraction of sequences, trees and graphs, BDDs, word and tree automata, pointer analysis);
– Case studies (abstractions used in ASTREE and TVLA);

Anticipated Home Work of Course 16.399: Abstract Interpretation

– A reading assignment of the slides for each course and of a recommanded recently published research article related to that course;
– A personnal project on the design and implementation of a static analyzer of numerical programs (which frontend will be provided)

Assigned reading for course 1

Patrick Cousot.
Abstract Interpretation Based Formal Methods and Future Challenges.
In *Informatics, 10 Years Back — 10 Years Ahead*, R. Wilhelm (Ed.), Lecture Notes in Computer Science 2000, pp. 138–156, 2001.

Anticipated Grading of Course 16.399: Abstract Interpretation

The course is letter graded.

10% Class participation
15% Presentation 1
15% Presentation 2
40% Personal project
20% Final written exam

The two presentations of research papers are in CS conference format (25mn of talk and 5mn of questions) to be selected by the students in the list of assigned readings; to be held outside of lecture hours — times TBA.

THE END

My MIT web site is http://www.mit.edu/~cousot/

The course web site is http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/.

