« Abstract Program Invariance and Termination Proofs »

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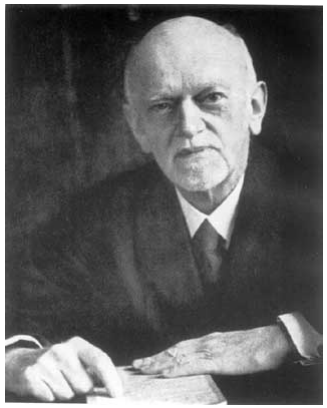
Undecidability

The fundamental limitation: undecidability

- In 1921, David Hilbert put forward the so-called Hilbert's Program, calling for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent, to be carried out using only "finitary" methods.
- Kurt Gödel's incompleteness theorems [1] essentially show that Hilbert's Program cannot be carried out.

Reference

[1] Gödel, K., "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte für Mathematik und Physik*, vol. 38 (1931), pp. 173–198.



David Hilbert

Kurt Gödel

Decision Problems

– A decision problem is a computational problem where the answer is always YES/NO:

$$\texttt{solve\_problem(data)} \mapsto \{\texttt{YES, NO}\}$$

– The complement $\neg P$ of a decision problem P is one where all the YES and NO answers are exchanged.

Decidability/Semidecidable/Undecidability

A decision problem is:

– "Decidable" if and only if there exists an algorithm to solve the problem in finite time;

– "Undecidable" if and only if there exists no algorithm to solve the problem in finite time;

– "Semidecidable" if and only if there exists an algorithm to solve the problem in finite time when the answer is `YES` but which may not terminate when the answer is `NO`;

The termination problem

– "Termination problem": given a sequential program and its input data, will execution of this program on these data ever terminate?

– The complement is the "Nontermination problem": given a sequential program and its input data, will execution of this program on these data never terminate?

– Termination is undecidable (but semidecidable);

– Nontermination is undecidable (and not semidecidable).

Interpretor

– We let `Ff` be the text file (of type `text`) containing the text of a program encoding a function f of type `text -> bool`;

– We let `Fd` be the text file (of type `text`) containing the text encoding the data d

– An interpreter `I : text * text -> bool` is a program which execution `I(Ff, Fd)` is the result $f(d)$ of the evaluation of function f on the data d.

Termination is semidecidable

`Terminaison(Ff, Fd) = if I(Ff, Fd) then YES else YES`

- Will answer `YES` if and only if `I(Ff, Fd)` that is $f(d)$ does terminate
- Will not terminate if and only if `I(Ff, Fd)` that is $f(d)$ does not terminate

Termination is undecidable

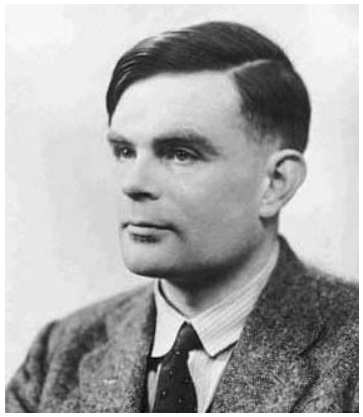
The proof given by Alonso Church [2] and Alan Turing [3] results from Gödel's second incompleteness theorem [1].

Reference

[2] Church, A., "An unsolvable problem of elementary number theory", *Fundamenta mathematicæ*, vol. 28, pp. 11–21, (1936).

[3] Turing, A.M., "Computability and λ-definability", *The Journal of Symbolic Logic*, vol. 2, pp. 153–163, (1937).

Alonso Church



Alan Turing

Proof that termination is undecidable

It can be *schematized* as follows:

- Assume, by reductio ad absurdum, that we can design a termination algorithm `T : text * text -> bool`, which execution is assumed to always terminate, and returns `T(Ff, Fd) = YES` if and only if `I(Ff, Fd)` terminates and `T(Ff, Fd) = NO` otherwise.

Let `Fc` be the text of a function `c : text -> bool` defined by:

```
c(F) = if T(F, F) then ¬I(F, F) else YES
```

Observe that execution of `c` always terminate, whence

```
T(Fc, Fc) = true
```

It follows that:

```
I(Fc, Fc) = if T(Fc, Fc) then ¬I(Fc, Fc) else YES
          = ¬I(Fc, Fc)
```

a contradiction.

Nontermination is not semidecidable

a) Decidable(P) $\Leftrightarrow$ [Semidecidable(P) $\wedge$ Semidecidable($\neg P$)]

$\Rightarrow$ obvious (by defining Semidecision(P) $\stackrel{\text{def}}{=}$ Decision(P) and Semidecision($\neg P$) $\stackrel{\text{def}}{=}\neg$Decision($P$));

$\Leftarrow$ alternatively execute one step of the semidecision algorithms of P and $\neg P$. Stop as soon as the first answer is returned.

b) ¬Semidecidable(¬Terminaison)

By reductio ad absurdum, Semidecidable(Terminaison) and Semidecidable(¬Terminaison) would imply Decidable(Terminaison).

Problem Reduction

– To prove that a problem P is undecidable, prove, by reductio ad absurdum, that if it were decidable, then the termination problem would be decidable.

Constant propagation is undecidable

– The constant propagation problem: determines whether, after initialization, a variable is constant (is never assigned a different value);
– The program `P` does not terminate if and only if the variable `X` has a constant value after initialization in the program:

```
var X : boolean; (* new variable not in P *)
X := true;
P;
X := false;
```


Absence of runtime errors is undecidable

The absence of runtime errors is not semidecidable:

– If absence of runtime errors in a program `P` where semi-decidable then the nontermination of `P` would be semi-decidable, by answering the question to know if `P; 1/0` has no runtime error.

What to do about undecidable problems?

Beyond simply abandoning:

– Consider decidable subcases only (but computational complexity strikes!)

– Ask for correct, intelligent, interactive human help

– Accept nontermination

– Accept approximations (*I don't know*)

Example of approximation: false alarms

When verifying the absence of runtime errors in a program, it may be the case that the automatic verifier is enable to establish statically that some error can be raised at runtime although this will never happen during execution.

For soundness, it must report a possibility of runtime error, which is impossible. This is called a false alarm.

Example of false alarms in ASTRÉE

```
% cat -n falsealarm.c
     1  /* falsealarm.c */
     2  void main()
     3  {
     4    int x, y;
     5    if ((-4681 < y) && (y < 4681) && (x < 32767) && (-32767 < x)
         && ((7*y*y - 1) == x*x)) {
     6      y = 1 / x;
     7    };
     8  }
% astree -exec-fn main falsealarm.c | grep WARN
falsealarm.c:6:9-6:14:[call#main@2:]: WARN: integer division by zero
                                              [-32766, 32766]
%
```

Computational Complexity

Computational Complexity

- Decidable problems can have a very high computational complexity;
- The time complexity of a problem is the number of steps that it takes to solve an instance of the problem, as a function of the size of the input, (usually measured in bits) using the most efficient algorithm;
- For example sorting an n-elements array is $\mathcal{O}(n \log n)$;

[1] Recall the big O notation: if $f(x)$ and $g(x)$ are two rean functions then $f(x)$ is $\mathcal{O}(g(x))$ as $x \to +\infty$ if and only if there exist numbers x_0 and M such that $|f(x)| \leq M|g(x)|$ for $x > x0$.

Complexity Classes

- the class P consists of all those decision problems that can be solved in polynomial time in the size of the input on a deterministic sequential machine;
- the class NP consists of all those decision problems whose positive solutions can be verified in polynomial time given the right information [1];
- the class co-NP consists of all those decision problems whose complement is in P.

Reference

[4] Hartmanis, J., and Stearns, R.E. " On the computational complexity of algorithms", Trans. Amer. Math. Soc. 117 (1965), 285–306.

[2] equivalently, whose solution can be found in polynomial time on a non-deterministic machine.

Problem Reduction

- A problem decision A is reducible [5] to a decision problem B

$\Longleftrightarrow$

- there exists a deterministic polynomial-time algorithm which transforms instances a of A into instances b of B, such that the answer to b is YES if and only if the answer to a is YES.

Reference

[5] R.M. Karp, "Reducibility among combinatorial problems", In *Complexity of Computer Computations*, R.E. Miller and J.W. Thatcher, editors, pages 85–103. Plenum Press, New York, NY, 1972.

NP-hardness

A decision problem is NP-hard if and only if
– every other problem in NP is reducible to it.

If we can find a polynomial algorithm to solve a NP-hard problem, then P = NP (?).

NP-completeness

A decision problem is NP-complete [6] if
– it is in NP, and
– every other problem in NP is reducible to it.

Reference

[6] Stephen A. Cook. "The Complexity of Theorem Proving Procedures". Proceedings Third Annual ACM Symposium on Theory of Computing (STOC), May 1971, pp 151–158.

The boolean satisfiability problem (SAT)

– An instance of SAT is defined by a Boolean expression written using only AND, OR, NOT, variables, and parentheses.
– The question is: given the expression, is there some assignment of TRUE and FALSE values to the variables that will make the entire expression true?
– For n variables, there are 2^n possible truth assignments to be checked.

Complexity of the boolean satisfiability problem

– The boolean satisfiability problem is NP-complete [7]

Reference

[7] Stephen A. Cook. "The Complexity of Theorem Proving Procedures". Proceedings Third Annual ACM Symposium on Theory of Computing (STOC), May 1971, pp 151–158.

What to do about NP-complete problems?

- Small is beautiful: Consider only problems of very small size.
- Special cases: An algorithm that is provably fast if the problem instances belong to a certain special case. Fixed-parameter algorithms can be seen as an implementation of this approach.
- Probabilistic: An algorithm that provably yields good average runtime behavior for a given distribution of the problem instances—ideally, one that assigns low probability to "hard" inputs.

- Heuristic: An algorithm that works "reasonably well" on many cases, but for which there is no proof that it is always fast (a rule of thumb, intuition).
- Approximation: An algorithm that quickly finds a suboptimal solution that is within a certain (known) range of the optimal one.

SAT solvers

- modern variants of the Davis-Logemann-Loveland algorithm [8] (depth first search with backtracking), such as zchaff [2] [9];
- stochastic local search algorithms, e.g. WalkSAT [10].

Reference

[8] M. Davis, G. Logemann and D. Loveland, "A Machine Program for Theorem-Proving", Communications of ACM, Vol. 5, No. 7, pp. 394-397, 1962

[9] M. Moskewicz, C. Madigan, Y. Zhao, L. Zhang, and S. Malik. "Chaff: Engineering an Efficient SAT Solver". 38th Design Automation Conference (DAC2001), Las Vegas, June 2001.

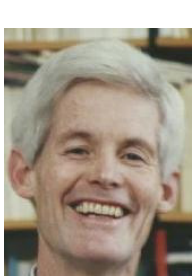
[10] Bart Selman, Henry Kautz, and Bram Cohen. "Local Search Strategies for Satisfiability Testing". in *Cliques, Coloring, and Satisfiability: Second DIMACS Implementation Challenge*, October 11-13, 1993. David S. Johnson and Michael A. Trick, ed. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 26, AMS, 1996.

[3] A few thousands variables, reported to solve a problem with a million variables and 10 million clauses.

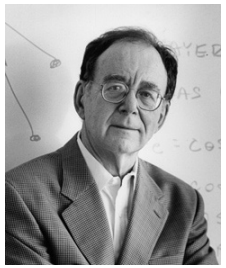
Polynomial Time Complexity

- Polynomial-time computability is identified with the intuitive notion of algorithmic efficiency;
- Intuitively valid only for small powers:

	Execution time at 10^9 ops/s			
n	$\mathcal{O}(n)$	$\mathcal{O}(n.log(n))$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$
1	ϵ	ϵ	ϵ	ϵ
10	ϵ	ϵ	0.1μs	1μs
10^3	1μs	6μs	1ms	1s
10^6	1ms	13ms	16mn	32 years
10^9	1s	20s	32 years	300 000 000 centuries
10^{12}	16mn	7.7h	300 000 centuries	—
10^{15}	11.6 days	1 year	—	—


S.A. Cook


R. Karp


J. Hartmanis


R.E. Stearn

Large problems

A computer program with

- 10 000 global variables
- represented on 32 bits
- and 500 000 lines of code

has about $2.10^{19} = 2^{64}$ states!

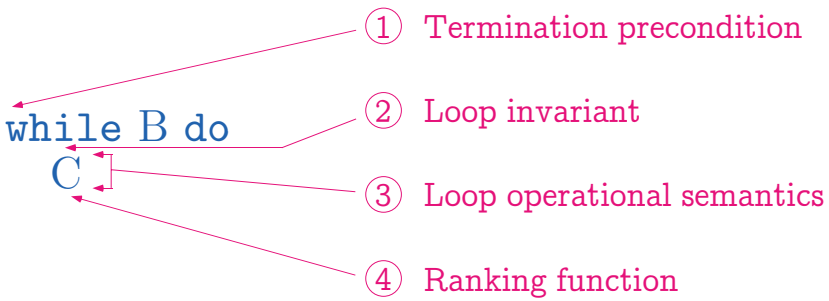
Abstract termination proofs

The following slides are based based on:

Patrick Cousot.
Proving Program Invariance and Termination by Parametric Abstraction, Lagrangian Relaxation and Semidefinite Programming (Invited paper).
In *Sixth International Conference on Verification, Model Checking and Abstract Interpretation (VMCAI'05)*.
Paris, France, January 17—19, 2005.
Lecture Notes in Computer Science 3385,
© Springer, Berlin, pp. 1—24.

Overview of the Termination Analysis Method

Proving Termination of a Loop



The main point in this talk is (4).

Proving Termination of a Loop

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
2. Assuming the *termination precondition*, perform an forward relational static analysis of the loop to determine the loop invariant
3. Assuming the loop invariant, perform an forward relational static analysis of the loop body to determine the loop abstract operational semantics
4. Assuming the loop semantics, use an abstraction of Floyd's ranking function method to prove termination of the loop

Arithmetic Mean Example

```
while (x <> y) do
    x := x - 1;
    y := y + 1
od
```

The polyhedral abstraction used for the static analysis of the examples is implemented using Bertrand Jeannet's NewPolka library.

Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
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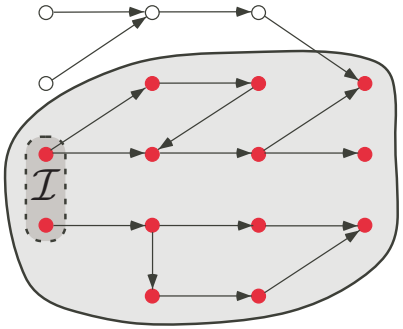
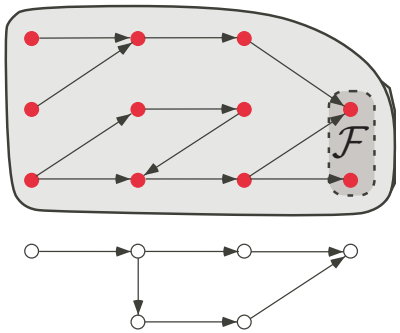
Forward/reachability properties



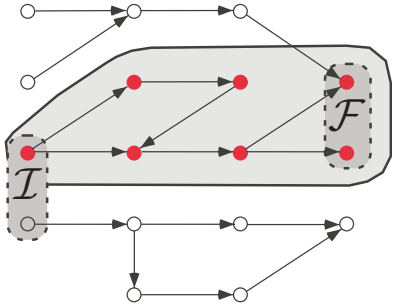
Example: partial correctness (must stay into safe states)

Backward/ancestry properties



Example: termination (must reach final states)

Forward/backward properties



Example: total correctness (stay safe while reaching final states)

Arithmetic Mean Example: Termination Precondition (1)

```
{x>=y}
  while (x <> y) do
    {x>=y+2}
      x := x - 1;
    {x>=y+1}
      y := y + 1
    {x>=y}
  od
{x=y}
```


Idea 1

The auxiliary termination counter method

Arithmetic Mean Example: Termination Precondition (2)

```
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

Add an auxiliary termination counter to enforce (bounded) termination in the backward analysis!

Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
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Arithmetic Mean Example: Loop Invariant

```
  assume ((x=y+2*k) & (x>=y));
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{k=0,x=y}
```


Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
2. Assuming the *termination precondition*, perform an forward relational static analysis of the loop to determine the loop invariant
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Arithmetic Mean Example: Body Relational Semantics

Case `x < y`:

```
assume (x=y+2*k)&(x>=y+2);
{x=y+2k,x>=y+2}
assume (x < y);
empty(6)
assume (x0=x)&(y0=y)&(k0=k);
empty(6)
k := k - 1;
x := x - 1;
y := y + 1
empty(6)
```

Case `x > y`:

```
assume (x=y+2*k)&(x>=y+2);
{x=y+2k,x>=y+2}
assume (x > y);
{x=y+2k,x>=y+2}
assume (x0=x)&(y0=y)&(k0=k);
{x=y+2k0,y=y0,x=x0,x=y+2k,
                    x>=y+2}
k := k - 1;
x := x - 1;
y := y + 1
{x+2=y+2k0,y=y0+1,x+1=x0,
                x=y+2k,x>=y}
```


Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
2. Assuming the *termination precondition*, perform an forward relational static analysis of the loop to determine the loop invariant
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4. Assuming the loop semantics, use an abstraction of Floyd's ranking function method to prove termination of the loop

Floyd's method for termination of `while B do C`

Given a loop invariant I, find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r such that:

– The rank is *nonnegative*:

$$\forall\, x_0, x : I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) \Rightarrow r(x_0) \geq 0$$

– The rank is *strictly decreasing*:

$$\forall\, x_0, x : I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) \Rightarrow r(x) \leq r(x_0) - \eta$$

$\eta \geq 1$ for $\mathbb{Z}$, $\eta > 0$ for $\mathbb{R}/\mathbb{Q}$ to avoid Zeno $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$...

Arithmetic Mean Example: Ranking Function

```
» clear all;
[v0,v] = variables('x','y','k')
% linear inequalities
%       x0 y0 k0
Ai =  [  0  0  0];
%        x  y  k
Ai_ = [  1 -1  0]; % x0 - y0 >= 0
bi = [0];
[N Mk(:,:,:)]=linToMk(Ai,Ai_,bi);
% linear equalities
%       x0 y0 k0
Ae =  [  0  0 -2;
         0 -1  0;
        -1  0  0;
         0  0  0];
%        x  y  k
Ae_ = [  1 -1  0;  % x - y - 2*k0 - 2 = 0
         0  1  0;  % y - y0 - 1 = 0
         1  0  0;  % x - x0 + 1 = 0
         1 -1 -2]; % x - y - 2*k = 0
be = [2; -1; 1; 0];
[M Mk(:,:,N+1:N+M)]=linToMk(Ae,Ae_,be);
```

Input the loop abstract semantics


```
» display_Mk(Mk, N, v0, v);
...
 +1.x -1.y >= 0
 -2.k0 +1.x -1.y +2 = 0
 -1.y0 +1.y -1 = 0
 -1.x0 +1.x +1 = 0
 +1.x -1.y -2.k = 0
...
```

– Display the abstract semantics of the loop `while B do C`

– compute ranking function, if any

```
» [diagnostic,R] = termination(v0, v, Mk, N, 'integer', 'linear');
» disp(diagnostic)
   feasible (bnb)
» intrank(R, v)
```

`r(x,y,k) = +4.k -2`

Proving Termination by Parametric Abstraction, Lagrangian Relaxation and Semidefinite Programming

Idea 2

Express the loop invariant and relational semantics as numerical positivity constraints

Relational semantics of `while B do C od` loops

- $x_0 \in \mathbb{R}/\mathbb{Q}/\mathbb{Z}$: values of the loop variables *before* a loop iteration
- $x \in \mathbb{R}/\mathbb{Q}/\mathbb{Z}$: values of the loop variables *after* a loop iteration
- $I(x_0)$: loop invariant, $[\![\mathtt{B};\mathtt{C}]\!](x_0, x)$: relational semantics of *one iteration of the loop body*
- $I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) = \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \quad (\geqslant_i \in \{>, \geq, =\})$
- not a restriction for numerical programs

Example of linear program (Arithmetic mean)

$[A\ A'][x_0\ x]^\top \geqslant b$

```
{x=y+2k,x>=y}
while (x <> y) do
    k := k - 1;
    x := x - 1;
    y := y + 1
od
```

```
+1.x -1.y >= 0
-2.k0 +1.x -1.y +2 = 0
-1.y0 +1.y -1 = 0
-1.x0 +1.x +1 = 0
+1.x -1.y -2.k = 0
```

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{array}\right] \left[\begin{array}{c} x_0 \\ y_0 \\ k_0 \\ x \\ y \\ k \end{array}\right] \begin{array}{c} \geq \\ = \\ = \\ = \\ = \end{array} \left[\begin{array}{c} 0 \\ -2 \\ 1 \\ -1 \\ 0 \end{array}\right]$$

Example of quadratic form program (factorial)

$[x\ x'] A [x\ x']^\top + 2[x\ x']\, q + r \geqslant 0$

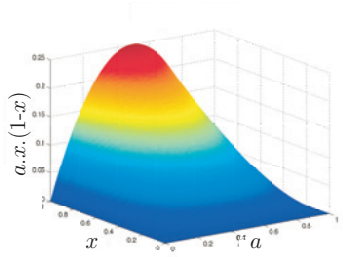
```
n := 0;
f := 1;
while (f <= N) do
    n := n + 1;
    f := n * f
od
```

```
-1.f0 +1.N0 >= 0
+1.n0 >= 0
+1.f0 -1 >= 0
-1.n0 +1.n -1 = 0
+1.N0 -1.N = 0
-1.f0.n +1.f = 0
```

$$[n_0 f_0 N_0 n f N] \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} n_0 \\ f_0 \\ N_0 \\ n \\ f \\ N \end{array}\right] + 2[n_0 f_0 N_0 n f N] \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{array}\right] + 0 = 0$$

Example of semialgebraic program (logistic map)

```
eps = 1.0e-9;
while (0 <= a) & (a <= 1 - eps)
       & (eps <= x) & (x <= 1) do
   x := a*x*(1-x)
od
```



Floyd's method for termination of `while B do C`

Find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r and $\eta > 0$ such that:

– The rank is *nonnegative*:

$$\forall\, x_0, x : \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \Rightarrow r(x_0) \geq 0$$

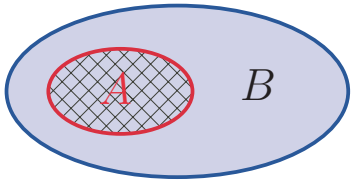
– The rank is *strictly decreasing*:

$$\forall\, x_0, x : \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \Rightarrow r(x_0) - r(x) - \eta \geq 0$$

Idea 3

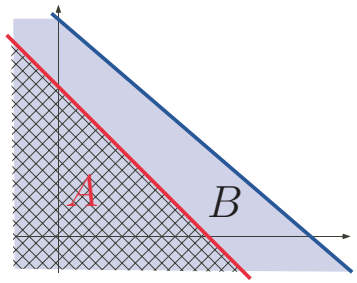
Eliminate the conjunction $\bigwedge$ and implication $\Rightarrow$ by Lagrangian relaxation

Implication (general case)



$$A \Rightarrow B$$
$$\Leftrightarrow$$
$$\forall x \in A : x \in B$$

Implication (linear case)



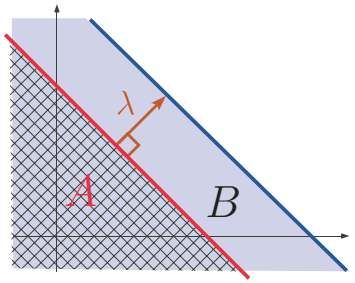
$A \Rightarrow B$ (assuming $A \neq \emptyset$)

$\Leftarrow$ (soundness)

$\Rightarrow$ (completeness)

border of A parallel to border of B

Lagrangian relaxation (linear case)



Lagrangian relaxation, formally

Let $\mathbb{V}$ be a finite dimensional linear vector space, $N > 0$ and $\forall k \in [0, N] : \sigma_k \in \mathbb{V} \mapsto \mathbb{R}$.

$$\forall x \in \mathbb{V} : \left(\bigwedge_{k=1}^{N} \sigma_k(x) \geq 0 \right) \Rightarrow (\sigma_0(x) \geq 0)$$

$\Leftarrow$ soundness (Lagrange)

$\Rightarrow$ completeness (*lossless*)

$\not\Rightarrow$ incompleteness (*lossy*)

$$\exists \lambda \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda_k \sigma_k(x) \geq 0$$

relaxation = approximation, λ_i = Lagrange coefficients

Lagrangian relaxation, equality constraints

$$\forall x \in \mathbb{V} : \left(\bigwedge_{k=1}^{N} \sigma_k(x) = 0 \right) \Rightarrow (\sigma_0(x) \geq 0)$$

$\Leftarrow$ soundness (Lagrange)

$$\exists \lambda \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda_k \sigma_k(x) \geq 0$$

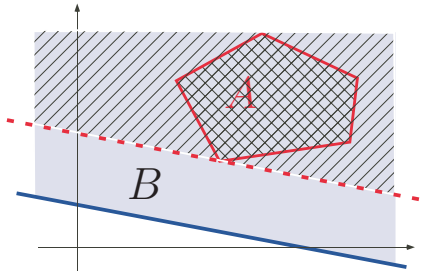
$$\wedge \ \exists \lambda' \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) + \sum_{k=1}^{N} \lambda'_k \sigma_k(x) \geq 0$$

$\Leftrightarrow (\lambda'' = \frac{\lambda' - \lambda}{2})$

$$\exists \lambda'' \in [1, N] \mapsto \mathbb{R} : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda''_k \sigma_k(x) \geq 0$$

Example: affine Farkas' lemma, informally

– An application of Lagrangian relaxation to the case when A is a polyhedron



Example: affine Farkas' lemma, formally

– Formally, if the system $Ax + b \geq 0$ is feasible then

$$\forall x : Ax + b \geq 0 \Rightarrow cx + d \geq 0$$

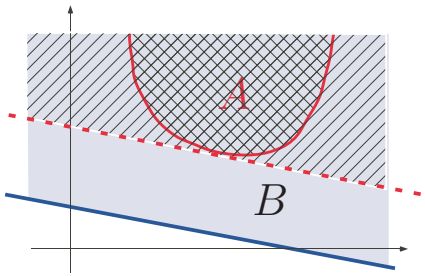
$\Leftarrow$ (soundness, Lagrange)

$\Rightarrow$ (completeness, Farkas)

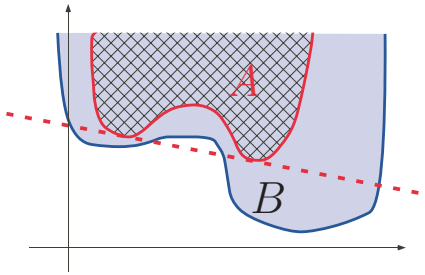
$$\exists \lambda \geq 0 : \forall x : cx + d - \lambda(Ax + b) \geq 0 \ .$$

Yakubovich's S-procedure, informally

– An application of Lagrangian relaxation to the case when A is a quadratic form



Incompleteness (convex case)



Yakubovich's S-procedure, completeness cases

– The constraint $\sigma(x) \geq 0$ is *regular* if and only if $\exists \xi \in \mathbb{V} : \sigma(\xi) > 0$.

– The S-procedure is lossless in the case of one regular quadratic constraint:

$$\forall x \in \mathbb{R}^n : x^\top P_1 x + 2q_1^\top x + r_1 \geq 0 \Rightarrow x^\top P_0 x + 2q_0^\top x + r_0 \geq 0$$

$\Leftarrow$ (Lagrange)

$\Rightarrow$ (Yakubovich)

$$\exists \lambda \geq 0 : \forall x \in \mathbb{R}^n : x^\top \left(\begin{bmatrix} P_0 & q_0 \\ q_0^\top & r_0 \end{bmatrix} - \lambda \begin{bmatrix} P_1 & q_1 \\ q_1^\top & r_1 \end{bmatrix} \right) x \geq 0.$$

Floyd's method for termination of `while B do C`

Find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r which is:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i} :$

$$\forall\ x_0, x : r(x_0) - \sum_{i=1}^{N} \lambda_i \sigma_i(x_0, x) \geq 0$$

– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i} :$

$$\forall\ x_0, x : (r(x_0) - r(x) - \eta) - \sum_{i=1}^{N} \lambda'_i \sigma_i(x_0, x) \geq 0$$

Idea 4

Parametric abstraction of the ranking function r

Parametric abstraction

– How can we compute the ranking function r?

$\rightarrow$ parametric abstraction:

1. Fix the form r_a of the function r a priori, in term of unkown parameters a
2. Compute the parameters a numerically

– Examples:

$r_a(x) = a.x^\top$	linear
$r_a(x) = a.(x\ 1)^\top$	affine
$r_a(x) = (x\ 1).a.(x\ 1)^\top$	quadratic

Floyd's method for termination of `while B do C`

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\, x_0, x : r_a(x_0) - \sum_{i=1}^{N} \lambda_i \sigma_i(x_0, x) \geq 0$$

– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\, x_0, x : (r_a(x_0) - r_a(x) - \eta) - \sum_{i=1}^{N} \lambda'_i \sigma_i(x_0, x) \geq 0$$

Idea 5

Eliminate the universal quantification $\forall$ using linear matrix inequalities (LMIs)

Mathematical programming

$$\exists x \in \mathbb{R}^n: \quad \bigwedge_{i=1}^{N} g_i(x) \geqslant 0$$

$$[\text{Minimizing} \quad f(x)]$$

feasibility problem : find a solution to the constraints

optimization problem : find a solution, minimizing $f(x)$

Example: Linear programming

$$\exists x \in \mathbb{R}^n: \quad Ax \geqslant b$$

$$[\text{Minimizing} \quad cx]$$

Feasibility

– feasibility problem: find a solution $s \in \mathbb{R}^n$ to the optimization program, such that $\bigwedge_{i=1}^{N} g_i(s) \geq 0$, or to determine that the problem is *infeasible*

– feasible set: $\{x \mid \bigwedge_{i=1}^{N} g_i(x) \geq 0\}$

– a feasibility problem can be converted into the optimization program

$$\min\{-y \in \mathbb{R} \mid \bigwedge_{i=1}^{N} g_i(x) - y \geq 0\}$$

Semidefinite programming

$$\exists x \in \mathbb{R}^n: \qquad M(x) \succcurlyeq 0$$

[Minimizing cx]

Where the linear matrix inequality (LMI) is

$$M(x) = M_0 + \sum_{k=1}^{n} x_k M_k$$

with symetric matrices ($M_k = {M_k}^\top$) and the positive semidefiniteness is

$$M(x) \succcurlyeq 0 = \forall X \in \mathbb{R}^N : X^\top M(x) X \geq 0$$

Semidefinite programming, once again

Feasibility is:

$$\exists x \in \mathbb{R}^n: \forall X \in \mathbb{R}^N : X^\top \left(M_0 + \sum_{k=1}^{n} x_k M_k \right) X \geq 0$$

of the form of the formulæ we are interested in for programs which semantics can be expressed as *LMIs*:

$$\bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 = \bigwedge_{i=1}^{N} (x_0\ x\ 1) M_i (x_0\ x\ 1)^\top \geqslant_i 0$$

Floyd's method for termination of `while B do C`

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i} :$

$$\forall\ x_0, x : r_a(x_0) - \sum_{i=1}^{N} \lambda_i (x_0\ x\ 1) M_i (x_0\ x\ 1)^\top \geq 0$$

– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i} :$

$$\forall\ x_0, x : (r_a(x_0) - r_a(x) - \eta) - \sum_{i=1}^{N} \lambda'_i (x_0\ x\ 1) M_i (x_0\ x\ 1)^\top \geq 0$$

Main steps in a typical soundness/completeness proof

$$\exists r : \forall x, x' : [\![B;C]\!](x\ x') \Rightarrow r(x, x') \geq 0$$

$$\Longleftrightarrow \exists r : \forall x, x' : \bigwedge_{k=1}^{N} \sigma_k(x, x') \geq 0 \Rightarrow r(x, x') \geq 0$$

$$\Longleftarrow \quad \text{⎧Lagrangian relaxation } (\Longrightarrow \text{ if lossless)⎫}$$

$$\exists r : \exists \lambda \in [1, N] \mapsto \mathbb{R}_* : \forall x, x' \in \mathbb{D}^n : r(x, x') - \sum_{k=1}^{N} \lambda_k \sigma_k(x\ x') \geq 0$$

$\Longleftarrow$ ⟅Semantics abstracted in LMI form ($\Longrightarrow$ if exact abstraction)⟆

$$\exists r : \exists \lambda \in [1, N] \mapsto \mathbb{R}_* : \forall x, x' \in \mathbb{D}^n : r(x, x') - \sum_{k=1}^{N} \lambda_k (x\ x'\ 1) M_k (x\ x'\ 1)^\top \geq 0$$

$\Longleftrightarrow$ ⟅Choose form of $r(x, x') = (x\ x'\ 1) M_0 (x\ x'\ 1)^\top$⟆

$$\Longleftrightarrow \exists M_0 : \exists \lambda \in [1, N] \mapsto \mathbb{R}_* : \forall x, x' \in \mathbb{D}^n : (x\ x'\ 1) M_0 (x\ x'\ 1)^\top - \sum_{k=1}^{N} \lambda_k (x\ x'\ 1) M_k (x\ x'\ 1)^\top \geq 0$$

$$\Longleftrightarrow \exists M_0 : \exists \lambda \in [1, N] \mapsto \mathbb{R}_* : \forall x, x' \in \mathbb{D}^{(n \times 1)} : \begin{bmatrix} x \\ x' \\ 1 \end{bmatrix}^\top \left(M_0 - \sum_{k=1}^{N} \lambda_k M_k \right) \begin{bmatrix} x \\ x' \\ 1 \end{bmatrix} \geq 0$$

$\Longleftrightarrow$ ⟅if $(x\ 1) A (x\ 1)^\top \geq 0$ for all x, this is the same as $(y\ t) A (y\ t)^\top \geq 0$ for all y and all $t \neq 0$ (multiply the original inequality by t^2 and call $xt = y$). Since the latter inequality holds true for all x and all $t \neq 0$, by continuity it holds true for all x, t, that is, the original inequality is equivalent to positive semidefiniteness of A⟆

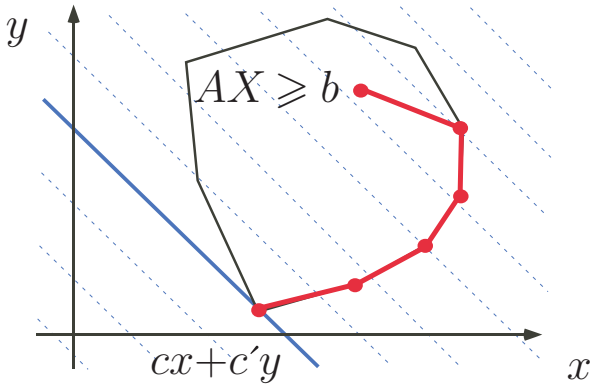
$$\exists M_0 : \exists \lambda \in [1, N] \mapsto \mathbb{R}_* : \left(M_0 - \sum_{k=1}^{N} \lambda_k M_k \right) \succcurlyeq 0$$

⟅LMI solver provides M_0 (and λ)⟆

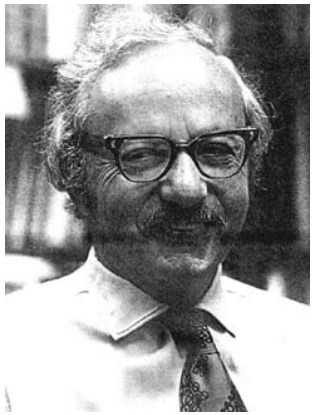
Idea 6

Solve the convex constraints by semidefinite programming

The simplex for linear programming



Dantzig 1948, exponential in worst case, good in practice



George Dantzig

Polynomial methods

Ellipsoid method : Khachian 1979 [11], polynomial in worst case but not good in practice

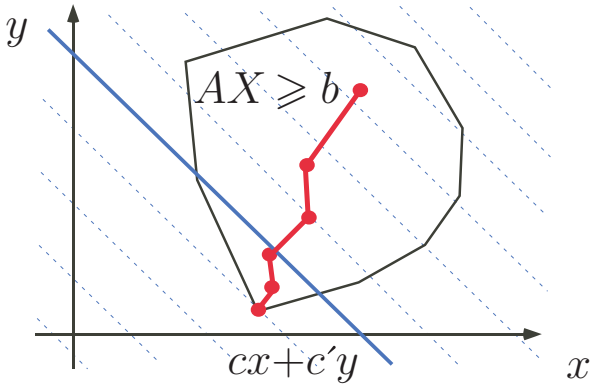
Interior point method : Narendra Karmarkar 1984 [12], polynomial in worst case and good in practice (hundreds of thousands of variables)

Reference

[11] L.G. Khachian. A polynomial algorithm in linear programming. Soviet Math. Dokl., 20:191-194, 1979.

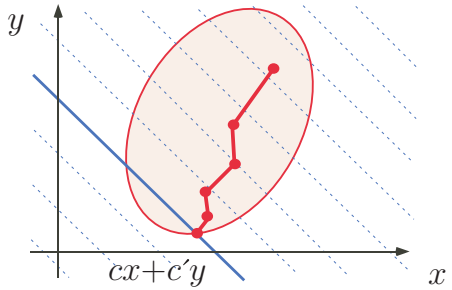
[12] Narendra Karmarkar. "A new polynomial-time algorithm for linear programming". Combinatorica 4(4): 373–396 (1984)

The interior point method



Interior point method for semidefinite programming

– Nesterov & Nemirovskii 1988 [13], polynomial in worst case and good in practice (thousands of variables)



– Various path strategies e.g. "stay in the middle"

Narendra Karmarkar

Arkadii Nemirovskii

Yurii Nesterov

Reference

[13] Yurii Nesterov and A. Nemirovsky. "Interior Point Polynomial Algorithms in Convex Programming" Society for Industrial and Applied Mathematics, 1994. (SIAM Studies in Applied Mathematics).

Semidefinite programming solvers

Numerous solvers available under MATHLAB®, a.o.:

– lmilab: P. Gahinet, A. Nemirovskii, A.J. Laub, M. Chilali

– Sdplr: S. Burer, R. Monteiro, C. Choi

– Sdpt3: R. Tütüncü, K. Toh, M. Todd

– SeDuMi: J. Sturm

– bnb: J. Löfberg (integer semidefinite programming)

Common interfaces to these solvers, a.o.:

– Yalmip: J. Löfberg

Sometime need some help (feasibility radius, shift,...)

Linear program: termination of Euclidean division

```
» clear all
% linear inequalities
%       y0 q0 r0
Ai =  [ 0  0  0; 0  0  0;
        0  0  0];
%        y  q  r
Ai_ = [ 1  0  0;  % y - 1 >= 0
        0  1  0;  % q - 1 >= 0
        0  0  1]; % r >= 0
bi = [-1; -1; 0];
% linear equalities
%       y0 q0 r0
Ae =  [ 0 -1  0;  % -q0 + q -1 = 0
       -1  0  0;  % -y0 + y = 0
        0  0 -1]; % -r0 + y + r = 0
%        y  q  r
Ae_ = [ 0  1  0; 1  0  0;
        1  0  1];
be = [-1; 0; 0];
```

Iterated forward/backward polyhedral analysis:

```
{y>=1}
q := 0;
{q=0,y>=1}
r := x;
{x=r,q=0,y>=1}
while (y <= r) do
   {y<=r,q>=0}
   r := r - y;
   {r>=0,q>=0}
   q := q + 1
   {r>=0,q>=1}
od
{q>=0,y>=r+1}
```

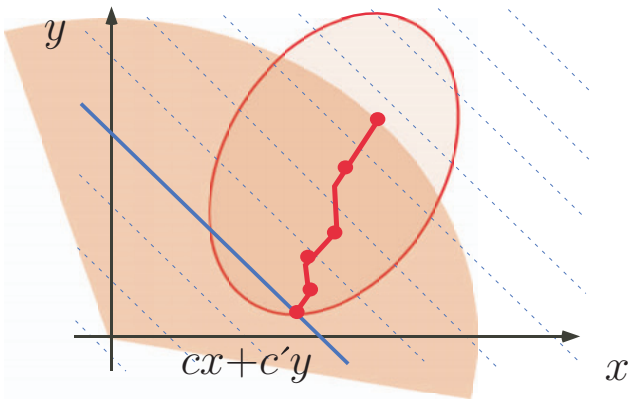


```
» [N Mk(:,:,:)]=linToMk(Ai, Ai_, bi);
» [M Mk(:,:,N+1:N+M)]=linToMk(Ae, Ae_, be);
» [v0,v]=variables('y','q','r');
» display_Mk(Mk, N, v0, v);
 +1.y -1 >= 0
 +1.q -1 >= 0
 +1.r >= 0
 -1.q0 +1.q -1 = 0
 -1.y0 +1.y = 0
 -1.r0 +1.y +1.r = 0
» [diagnostic,R] = termination(v0, v, Mk, N, 'integer', 'quadratic');
» disp(diagnostic)
   termination (bnb)
» intrank(R, v)
```

```
r(y,q,r) = -2.y +2.q +6.r
```

Floyd's proposal $r(x, y, q, r) = x - q$ is more intuitive but requires to discover the nonlinear loop invariant $x = r + qy$.

Imposing a feasibility radius



Quadratic program: termination of factorial

Program:

```
n := 0;
f := 1;
while (f <= N) do
    n := n + 1;
    f := n * f
od
```

LMI semantics:

```
-1.f0 +1.N0 >= 0
+1.n0 >= 0
+1.f0 -1 >= 0
-1.n0 +1.n -1 = 0
+1.N0 -1.N = 0
-1.f0.n +1.f = 0
```

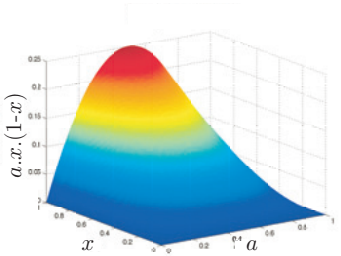
```
r(n,f,N) = -9.993455e-01.n +4.346533e-04.f
           +2.689218e+02.N +8.744670e+02
```


Idea 7

Convex abstraction of non-convex constraints

Semidefinite programming relaxation for polynomial programs

```
eps = 1.0e-9;
while (0 <= a) & (a <= 1 - eps)
       & (eps <= x) & (x <= 1) do
   x := a*x*(1-x)
od
```



Write the verification conditions in polynomial form, use SOS solver to relax in semidefinite programming form.

SOStool+SeDuMi:

```
r(x) = 1.222356e-13.x + 1.406392e+00
```


Principle

- Show $\forall x : p(x) \geq 0$ by $\forall x : p(x) = \sum_{i=1}^{k} q_i(x)^2$
- Hibert's 17th problem (sum of squares)
- Undecidable (but for monovariable or low degrees)
- Look for an approximation (relaxation) by semidefinite programming

General relaxation/approximation idea

- Write the polynomials in quadratic form with monomials as variables: $p(x, y, \ldots) = z^\top Q z$ where $Q \succcurlyeq 0$ is a semidefinite positive matrix of unknowns and $z = [\ldots x^2, xy, y^2, \ldots x, y, \ldots 1]$ is a monomial basis
- If such a Q does exist then $p(x, y, \ldots)$ is a sum of squares [3]
- The equality $p(x, y, \ldots) = z^\top Q z$ yields LMI contrains on the unkown Q: $z^\top M(Q) z \succcurlyeq 0$

[4] Since $Q \succcurlyeq 0$, Q has a Cholesky decomposition L which is an upper triangular matrix L such that $Q{=}L^\top L$. It follows that $p(x) = z^\top Q z = z^\top L^\top L z = (Lz)^\top Lz = [L_{i,:} \cdot z]^\top [L_{i,:} \cdot z] = \sum_i (L_{i,:} \cdot z)^2$ (where $\cdot$ is the vector dot product $x \cdot y = \sum_i x_i y_i$), proving that $p(x)$ is a sum of squares whence $\forall x : p(x) \geq 0$, which eliminates the universal quantification on x.

- Instead of quantifying over monomials values x, y, replace the monomial basis z by auxiliary variables X (loosing relationships between values of monomials)
- To find such a $Q \succcurlyeq 0$, check for semidefinite positiveness $\exists Q : \forall X : X^\top M(Q) X \geq 0$ i.e. $\exists Q : M(Q) \succcurlyeq 0$ with LMI solver
- Implement with SOStools under MATHLAB® of Prajna, Papachristodoulou, Seiler and Parrilo
- Nonlinear cost since the monomial basis has size $\binom{n+m}{m}$ for multivariate polynomials of degree n with m variables

Considering More General Forms of Programs

Handling disjunctive loop tests and tests in loop body

- By case analysis
- and "conditional Lagrangian relaxation" (Lagrangian relaxation in each of the cases)

Loop body with tests

```
while (x < y) do
  if (i >= 0) then
     x := x+i+1
  else
     y := y+i
  fi
od
```

$\longrightarrow$ case analysis: $\begin{cases} \mathtt{i} \geq 0 \\ \mathtt{i} < 0 \end{cases}$

lmilab:

```
r(i,x,y) = -2.252791e-09.i -4.355697e+07.x +4.355697e+07.y
                                            +5.502903e+08
```


Quadratic termination of linear loop

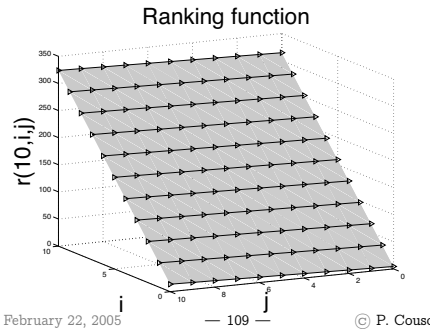
```
{n>=0}
i := n; j := n;
while (i <> 0) do
  if (j > 0) then
    j := j - 1
  else
    j := n; i := i - 1
  fi
od
```

$\longleftarrow$ termination precondition determined by iterated forward/backward polyhedral analysis

`sdplr` (with feasibility radius of `1.0e+3`):

```
r(n,i,j) = +7.024176e-04.n^2 +4.394909e-05.n.i ...
           -2.809222e-03.n.j +1.533829e-02.n ...
           +1.569773e-03.i^2 +7.077127e-05.i.j   ...
           +3.093629e+01.i -7.021870e-04.j^2 ...
           +9.940151e-01.j +4.237694e+00
```

Successive values of $r(n, i, j)$ for $n = 10$ on loop entry



Handling nested loops

- by induction on the loop depth
- use an iterated forward/backward symbolic analysis to get a necessary termination precondition
- use a forward symbolic symbolic analysis to get the semantics of a loop body
- use Lagrangian relaxation and semidefinite programming to get the ranking function

Example of termination of nested loops: Bubblesort inner loop

```
...
+1.i' -1 >= 0
+1.j' -1 >= 0
+1.n0' -1.i' >= 0
-1.j +1.j' -1 = 0
-1.i +1.i' = 0
-1.n +1.n0' = 0
+1.n0 -1.n0' = 0
+1.n0' -1.n' = 0
...
```

Iterated forward/backward polyhedral analysis followed by forward analysis of the body:

```
assume (n0 = n & j >= 0 & i >= 1 & n0 >= i & j <> i);
{n0=n,i>=1,j>=0,n0>=i}
assume (n01 = n0 & n1 = n & i1 = i & j1 = j);
{j=j1,i=i1,n0=n1,n0=n01,n0=n,i>=1,j>=0,n0>=i}
j := j + 1
{j=j1+1,i=i1,n0=n1,n0=n01,n0=n,i>=1,j>=1,n0>=i}
termination (lmilab)
r(n0,n,i,j) = +434297566.n0 +226687644.n -72551842.i
                                          -2.j +2147483647
```


Example of termination of nested loops: Bubblesort outer loop

```
...
+1.i' +1 >= 0
+1.n0' -1.i' -1 >= 0
+1.i' -1.j' +1 = 0
-1.i +1.i' +1 = 0
-1.n +1.n0' = 0
+1.n0 -1.n0' = 0
+1.n0' -1.n' = 0
...
```

Iterated forward/backward polyhedral analysis followed by forward analysis of the body:

```
  assume (n0=n & i>=0 & n>=i & i <> 0);
{n0=n,i>=0,n0>=i}
  assume (n01=n0 & n1=n & i1=i & j1=j);
{j1=j,i=i1,n0=n1,n0=n01,n0=n,i>=0,n0>=i}
  j := 0;
  while (j <> i) do
      j := j + 1
  od;
  i := i - 1
{i+1=j,i+1=i1,n0=n1,n0=n01,n0=n,i+1>=0,n0>=i+1}
termination (lmilab)
r(n0,n,i,j) = +24348786.n0 +16834142.n +100314562.i +65646865
```

Handling nondeterminacy

- By case analysis
- Same for concurrency by interleaving
- Same with fairness by nondeterministic interleaving with encoding of an explicit scheduler

Termination of a concurrent program

```
[|  1: while [x+2 < y] do
    2:     [x := x + 1]
       od
    3:
||
    1: while [x+2 < y] do
    2:     [y := y - 1]
       od
    3:
|]
```

interleaving ⟶

```
while (x+2 < y) do
   if ?=0 then
     x := x + 1
   else if ?=0 then
     y := y - 1
   else
     x := x + 1;
     y := y - 1
   fi fi
od
```

penbmi: `r(x,y) = 2.537395e+00.x+-2.537395e+00.y+ -2.046610e-01`

Termination of a fair parallel program

```
[[ while [(x>0)|(y>0) do x := x - 1] od ||
   while [(x>0)|(y>0) do y := y - 1] od ]]
```

interleaving + scheduler ⟶

{m>=1} ← termination precondition determined by iterated forward/backward polyhedral analysis

```
t := ?;
assume (0 <= t & t <= 1);
s := ?;
assume ((1 <= s) & (s <= m));
while ((x > 0) | (y > 0)) do
  if (t = 1) then
     x := x - 1
  else
     y := y - 1
  fi;
  s := s - 1;
  if (s = 0) then
    if (t = 1) then
       t := 0
    else
       t := 1
    fi;
    s := ?;
    assume ((1 <= s) & (s <= m))
  else
    skip
  fi
od;;
```

penbmi: `r(x,y,m,s,t) = +1.000468e+00.x +1.000611e+00.y +2.855769e-02.m -3.929197e-07.s +6.588027e-06.t +9.998392e+03`

Relaxed Parametric Invariance Proof Method

Floyd's method for invariance

Given a loop precondition P, find an unkown loop invariant I such that:

– The invariant is *initial*:

$$\forall\, x : P(x) \Rightarrow I(x)$$
$$\uparrow$$

– The invariant is *inductive*:

$$\forall\, x, x' : I(x) \wedge [\![\mathtt{B};\mathtt{C}]\!](x, x') \Rightarrow I(x')$$
$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$
$$???$$

Abstraction

– Express loop semantics as a conjunction of LMI constraints (by relaxation for polynomial semantics)

– Eliminate the conjunction and implication by Lagrangian relaxation

– Fix the form of the unkown invariant by parametric abstraction

. . . we get . . .

Floyd's method for numerical programs

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

– The invariant is *initial*: $\exists \mu \in \mathbb{R}^+ :$

$$\forall\, x : I_a(x) - \mu.P(x) \geq 0$$

– The invariant is *inductive*: $\exists \lambda \in [0, N] \longrightarrow \mathbb{R}^+ :$

$$\forall\, x, x' : I_a(x') - \lambda_0.I_a(x) - \sum_{k=1}^{N} \lambda_k.\sigma_k(x, x') \geq 0$$
$$\uparrow \quad \uparrow$$
$$\text{bilinear in } \lambda_0 \text{ and } a$$

Idea 8

Solve the bilinear matrix inequality (BMI) by semidefinite programming

Bilinear matrix inequality (BMI) solvers

$$\exists x \in \mathbb{R}^n : \bigwedge_{i=1}^{m} \left(M_0^i + \sum_{k=1}^{n} x_k M_k^i + \sum_{k=1}^{n} \sum_{\ell=1}^{n} x_k x_\ell N_{k\ell}^i \succcurlyeq 0 \right)$$

[Minimizing $x^\top Q x + cx$]

Two solvers available under MATHLAB®:

- PenBMI: M. Kočvara, M. Stingl
- bmibnb: J. Löfberg

Common interfaces to these solvers:

- Yalmip: J. Löfberg

Example: linear invariant

Program:

```
i := 2; j := 0;
while (??) do
  if (??) then
    i := i + 4
  else
    i := i + 2;
    j := j + 1
  fi
od;
```

- Invariant:

```
+2.14678e-12*i -3.12793e-10*j +0.486712 >= 0
```

- Less natural than $i - 2j - 2 \geq 0$
- Alternative:
  - Determine parameters (a) by other methods (e.g. random interpretation)
  - Use BMI solvers to *check* for invariance

Conclusion

Constraint resolution failure

- infeasibility of the constraints does not mean "non termination" or "non invariance" but simply failure
- inherent to abstraction!

Numerical errors

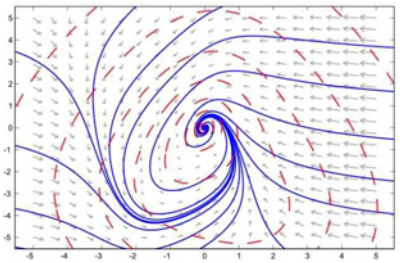
- LMI/BMI solvers do numerical computations with rounding errors, shifts, etc
- ranking function is subject to numerical errors
- the hard point is to discover a candidate for the ranking function
- much less difficult, when the ranking function is known, to re-check for satisfaction (e.g. by static analysis)
- not very satisfactory for invariance (checking only ???)

Related work

- Linear case (Farkas lemma):
  - Invariants: Sankaranarayanan, Spima, Manna (CAV'03, SAS'04, heuristic solver)
  - Termination: Podelski & Rybalchenko (VMCAI'03, Lagrange coefficients eliminated by hand to reduce to linear programming so no disjunctions, no tests, etc)
  - Parallelization & scheduling: Feautrier, easily generalizable to nonlinear case

Seminal work

- LMI case, Lyapunov 1890, *"an invariant set of a differential equation is stable in the sense that it attracts all solutions if one can find a function that is bounded from below and decreases along all solutions outside the invariant set"*.



THE END, THANK YOU

