« Mathematical foundations: (6) Abstraction — Part II »

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THESE

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par

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METHODES ITERATIVES DE CONSTRUCTION ET D'APPROXIMATION DE PCINTS FIXES D'OPERATEURS MONOTONES SUR UN TREILLIS, ANALYSE SEMANTIQUE DES PROGRAMMES.

Thèse soutenue le 21 mars 1978 devant la Commission d'Examen :

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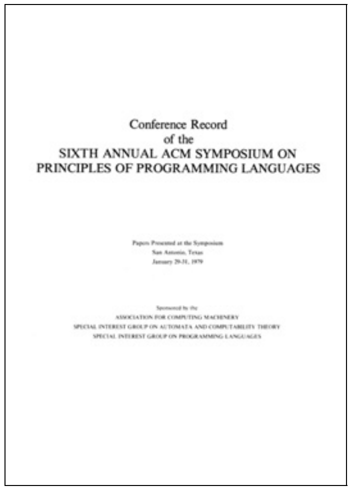
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Exact fixpoint abstraction

Property transformer abstraction

Let

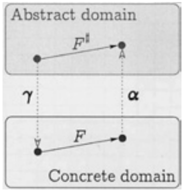
- $\langle L, \leq, 0, 1, \wedge, \vee\rangle$ be complete lattice
- $F \in L \stackrel{\text{m}}{\longmapsto} L$ be a monotonic transfer fonction
- $\alpha \in L \mapsto \overline{L}$ be an abstraction

We would like to:

- compute $\alpha(\mathsf{lfp}\, F)$
- without computing $\mathsf{lfp}\, F$ (which is, in general, impossible)

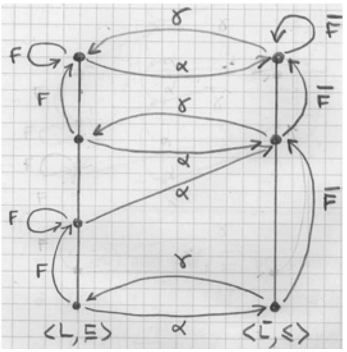
One solution is to have:

- an abstract transformer $\overline{F} \in \overline{L} \mapsto \overline{L}$
- such that $\alpha(\mathsf{lfp}\, F) = \mathsf{lfp}\, \overline{F}$
- to exclude trivial solutions (like $\overline{F} = \lambda X \cdot \alpha(\mathsf{lfp}\, F)$), more must be imposed on the choice of $\overline{F}$
- for monotonic functions, one way may be to use higher-order abstraction



$$\langle P, \subseteq\rangle \langle \alpha, \gamma\rangle \langle Q, \sqsubseteq\rangle$$
$$\Longrightarrow$$
$$\langle P \overset{\mathrm{m}}{\longmapsto} P, \dot{\subseteq}\rangle \underset{\lambda F \cdot \alpha \circ F \circ \gamma}{\overset{\lambda \overline{F} \cdot \gamma \circ \overline{F} \circ \alpha}{\leftrightarrows}} \langle Q \overset{\mathrm{m}}{\longmapsto} Q, \dot{\sqsubseteq}\rangle$$

- In general however, $\langle L, \leq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{L}, \sqsubseteq\rangle$, $F \in L \overset{\mathrm{m}}{\longmapsto} L$ and $\overline{F} = \alpha \circ F \circ \gamma$ does not imply $\alpha(\mathsf{lfp}^{\leq} F) = \mathsf{lfp}^{\sqsubseteq} \overline{F}$ as shown by the following counter-example:

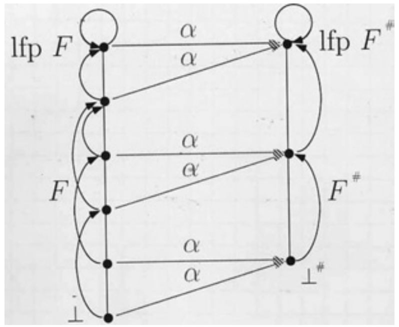


So we are interested in studying fixpoint abstraction:

- Exact abstraction: $\alpha(\mathsf{lfp}\, F) = \mathsf{lfp}\, \overline{F}$
- Approximate abstraction: $\alpha(\mathsf{lfp}\, F) \sqsubset \mathsf{lfp}\, \overline{F}$

À la Kleene exact fixpoint abstraction with Galois connections [1]

Example:



[1] also called fixpoint fusion, fixpoint inducing, lifting, morphism, transfer, precise abstraction, etc…

THEOREM. If

- $\langle L, \leq\rangle$ and $\langle M, \sqsubseteq\rangle$ are posets;
- $\langle L, \leq\rangle \xrightleftharpoons[\alpha]{\gamma} \langle M, \sqsubseteq\rangle$ and $F^\sharp \in M \mapsto M$

or $\langle L, \leq\rangle \xrightleftharpoons[\alpha]{\gamma} \langle M, \sqsubseteq\rangle$ and $F^\sharp \in M \stackrel{m}{\longmapsto} M$

- $F \in L \stackrel{m}{\not\longmapsto} L$ is a monotonic partial map
- $\perp \leq F(\perp)$
- $F^0 = \perp$, $F^{\delta+1} = F(F^\delta)$, $\bigvee_{\beta<\lambda} F^\beta$ exists when λ is a limit ordinal and $F^\lambda = \bigvee_{\beta<\lambda} F^\beta$
- $\forall \delta : \alpha \circ F(F^\delta) = F^\sharp \circ \alpha(F^\delta)$

then:

- $F^\sharp = \alpha \circ F \circ \gamma$
- $\alpha(\mathsf{lfp}_\perp^{\leq} F) = \mathsf{lfp}_{\alpha\perp}^{\sqsubseteq} F^\sharp$
- The iteration order of $F^\sharp$ is less than or equal to that of F

■

PROOF.

Lemma 1

$\forall \delta \in \mathbb{O} : \alpha(F^\delta) = F^{\sharp\delta}$ where $F^{\sharp 0} = \perp^\sharp \stackrel{\text{def}}{=} \alpha(\perp)$, $F^{\sharp\delta+1} = F^\sharp(F^{\sharp\delta})$ and $F^{\sharp\lambda} = \bigsqcup_{\beta<\lambda} F^{\sharp\beta}$ is well-defined when λ is a limit ordinal.

PROOF. By transfinite induction on δ:

- $\alpha(\perp) = \perp^\sharp$, — def. $\perp^\sharp$
- The induction hyp. $\alpha(F^\delta) = F^{\sharp\delta}$ implies

$\alpha(F^{\delta+1})$	
$= \alpha(F(F^\delta))$	def. $F^{\sharp\delta+1}$
$= F^\sharp(\alpha(F^\delta))$	commutation property
$= F^\sharp(F^{\sharp\delta})$	ind. hyp.
$= F^{\sharp\delta+1}$	def. $F^{\sharp\delta}$

- If $\alpha(F^\beta) = F^{\sharp\beta}$ for $\beta < \lambda$, λ limit ordinal by induction hyp., then

$\alpha(F^\lambda)$	
$= \alpha(\bigvee_{\beta<\lambda} F^\beta)$	def. F^λ where the lub is assumed to exist
$= \bigsqcup_{\beta<\lambda} \alpha(F^\beta)$	since α preserves existing lubs so $\bigsqcup_{\beta<\lambda} \alpha(F^\beta)$ exists
$= \bigsqcup_{\beta<\lambda} F^{\sharp\beta}$	induction hyp.
$= F^{\sharp\lambda}$	def. $F^{\sharp\lambda}$ for λ limit ordinal.

□

Lemma 2

$F^\delta, \delta \in \mathbb{O}$ is a $\sqsubseteq$-increasing chain.

PROOF. By transfinite induction on δ:

- $F^0 = \perp \leq F(\perp) = F^1$ — since $\perp$ is assumed to be a prefixpoint
- If $F^\delta \leq F^{\delta+1}$ — induction hyp.
  $\Longrightarrow F^{\delta+1} = F(F^\delta) \leq F(F^{\delta+1}) = F^{\delta+2}$ — by monotony
- $\forall \delta < \lambda$, λ limit ordinal:
  $F^\delta \leq \bigsqcup_{\beta<\lambda} F^\beta = F^\lambda$ — by def. of lubs and F^δ assumed to exist.

□

Lemma 3

$$\exists \epsilon \in \mathbb{O} : F(F^{\epsilon}) = F^{\epsilon}.$$

PROOF. – The chain $F^{\delta}, \delta \in \mathbb{O}$ cannot be strictly increasing since it is included in the set L so that its cardinality must be less than that of L proving that $\exists \epsilon < \epsilon' : F^{\epsilon} = F^{\epsilon'}$ where ϵ' is the least ordinal with the same cardinality as L.

– $F^{\epsilon} \leq F^{\epsilon+1} \leq F^{\epsilon'} = F^{\epsilon}$ so that $F^{\epsilon} = F^{\epsilon+1} = F(F^{\epsilon})$ by antisymmetry and definition of the iterates. $\square$

Lemma 4

$$\text{If } \bot \leq X = F(X) \text{ then } \forall \delta \in \mathbb{O} : F^{\delta} \leq X.$$

PROOF. By transfinite induction on δ:

– $F^0 = \bot \leq X$ by hypothesis;

– If $F^{\delta} \leq X$ by induction hyp., then

$F^{\delta+1} = F(F^{\delta}) \leq F(X) = X$ by ind. hyp., monotonicity and fixpoint property;

– if $F^{\beta} \leq X$ for all $\beta < \lambda$, λ limit ordinal, then: $F^{\delta} = \bigsqcup_{\beta<\lambda} F^{\beta} \leq X$ by def. of lub (assumed to exist). $\square$

Lemma 5

$$\text{If } \exists \epsilon \in \mathbb{O} : F^{\epsilon} = \mathsf{lfp}^{\leq}_{\bot} F.$$

PROOF. By lemma 2, $\bot = F^0 \leq F^{\epsilon}$ and $F(F^{\epsilon}) = F^{\epsilon}$ by lemma 3.

If $\bot \leq X = F(X)$ then, by lemma 4, $F^{\epsilon} \leq X$ proving that F^{ϵ} is the least fixpoint of F greater than or equal to $\bot$, so $F^{\epsilon} = \mathsf{lfp}^{\leq}_{\bot} F$. $\square$

Lemma 6

$$\text{If } F^{\sharp^{\epsilon}} = F^{\sharp}(F^{\sharp^{\epsilon}}).$$

PROOF. $F^{\sharp^{\epsilon+1}} = F^{\sharp}(F^{\sharp^{\epsilon}}) = F^{\sharp}(\alpha(F^{\epsilon})) = \alpha(F(F^{\epsilon})) = \alpha(F^{\epsilon}) = F^{\sharp^{\epsilon}}$ by def. $F^{\sharp^{\epsilon+1}}$ =, lemma 1, commutation property, lemma 3, lemma 1. $F^{\sharp^{\epsilon}}$ exists by lemma 1. $\square$

Lemma 7

$F^{\sharp}$ is monotonic.

PROOF.

$\alpha \circ F = F^{\sharp} \circ \alpha$ — commutation property

$\Longrightarrow \lambda x \cdot \alpha \circ F \circ \gamma(x) = F^{\sharp} \circ \alpha \circ \gamma(x)$ — def. = on functions

$\Longrightarrow \alpha \circ F \circ \gamma = F^{\sharp}$ since $\alpha \circ \gamma(x) = x$ when α is surjective in a Galois connection

$\Longrightarrow F^{\sharp}$ is monotonic, as a composition of monotonic functions. $\square$

Note: Instead of α surjective and $F^{\sharp} \in M \mapsto M$, we can also assume that $F^{\sharp} \in M \overset{\text{m}}{\longmapsto} M$ without assuming α surjective.

Lemma 8

$$\text{If } \alpha(\bot) \sqsubseteq Y = F^{\sharp}(Y) \text{ then } \forall \delta \in \mathbb{O} : F^{\sharp^{\delta}} \sqsubseteq Y.$$

PROOF. Identical to that of lemma 4, using lemma 7 (or the corresponding hypothesis $F^{\sharp} \in M \overset{\text{m}}{\longmapsto} M$). $\square$

Lemma 9

$$F^{\sharp^{\epsilon}} = \mathsf{lfp}^{\sqsubseteq}_{\alpha\bot} F^{\sharp}.$$

PROOF. $\bot \leq F^{\epsilon}$ hence $\alpha(\bot) \sqsubseteq \alpha(F^{\epsilon}) = F^{\sharp^{\epsilon}}$ by lemma 2, α is monotonic, lemma 1. We have $F^{\sharp}(F^{\sharp^{\epsilon}}) = F^{\sharp^{\epsilon}}$ by lemma 6. If $\alpha(\bot) \sqsubseteq Y = F^{\sharp}(Y)$ then $F^{\sharp^{\epsilon}} \sqsubseteq Y$ by lemma 8. $\square$

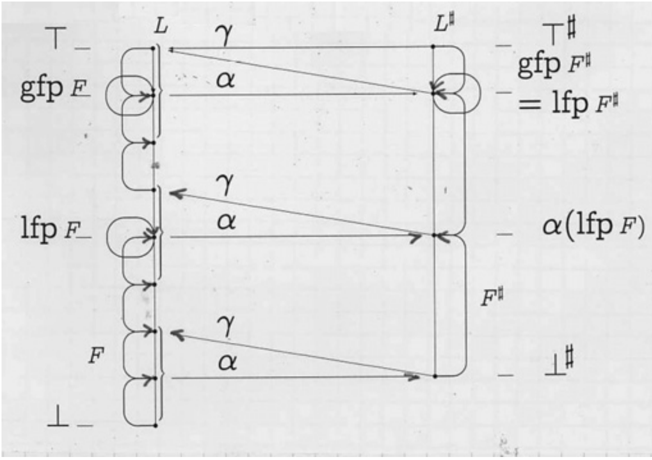
PROOF. (of the theorem)

$$\begin{array}{lll} & \alpha(\mathsf{lfp}^{\leq}_{\bot} F) & \\ = & \alpha(F^{\epsilon}) & \text{by lemma 5} \\ = & F^{\sharp^{\epsilon}} & \text{by lemma 1} \\ = & \mathsf{lfp}^{\sqsubseteq}_{\alpha\bot} F^{\sharp} & \text{by lemma 9} \end{array}$$

$\square$

$\square$

example:



À la Kleene exact fixpoint abstraction with continuous abstraction

THEOREM. If

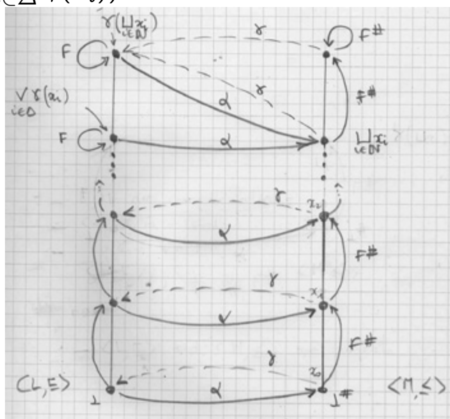
- $\langle L, \leq\rangle$ and $\langle M, \sqsubseteq\rangle$ are posets
- $F \in L \overset{\text{m}}{\longmapsto} L$ is a monotonic partial map on L
- $F^\sharp \in M \overset{\text{m}}{\longmapsto} M$ is a monotonic map on M
- $\alpha \in L \mapsto M$ is upper continuous
- $\perp \leq F(\perp)$
- $F^0 = \perp$, $F^{\delta+1} = F(F^\delta)$, $\bigvee_{\beta<\lambda} F^\beta$ exists when λ is a limit ordinal in which case $F^\lambda = \bigvee_{\beta<\lambda} F^\beta$
- $\forall \delta \in \mathbb{O} : \alpha \circ F(F^\delta) = F^\sharp \circ \alpha(F^\delta)$

then

- $\alpha(\mathsf{lfp}^{\leq}_{\perp} F) = \mathsf{lfp}^{\sqsubseteq}_{\alpha(\perp)} F^\sharp$

PROOF. Essentially identical to that of the previous theorem. One first prove lemma 2 by monotony of F and then lemma 1 since α is upper continuous and the iterates pf F from $\perp$ form an increasing chain. The proofs of lemmata 3 to 6 and 8 to 9 are unchanged while lemma 7 is now an hypothesis. The rpoof of the theorem is identical. ☐

Note: the commutation condition does not work with γ for $\mathsf{lfp}\ F$ (it does, by duality, for $\mathsf{gfp}\ F$). A counter-example is $(\gamma(\bigsqcup_{i\in\Delta} x_i) \neq \bigvee_{i\in\Delta} \gamma(x_i))$:



We have

- $\langle L, \leq\rangle$ and $\langle M, \sqsubseteq\rangle$ are posets
- F, $F^\sharp$ are monotone
- $\langle L, \leq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle L, \sqsubseteq\rangle$
- $\gamma \circ F^\sharp = F \circ \gamma$

but

- $\alpha(\mathsf{lfp}\ F) \neq \mathsf{lfp}\ F^\sharp$

À la Tarski exact fixpoint abstraction

THEOREM. If $\langle \mathcal{D}^\natural, \sqsubseteq^\natural, \perp^\natural, \sqcup^\natural \rangle$ and $\langle \mathcal{D}^\sharp, \sqsubseteq^\sharp, \perp^\sharp, \sqcup^\sharp \rangle$ are complete lattices, $F^\natural \in \mathcal{D}^\natural \xrightarrow{m} \mathcal{D}^\natural$, $F^\sharp \in \mathcal{D}^\sharp \xrightarrow{m} \mathcal{D}^\sharp$ are monotonic and

- α is a complete $\sqcap$-morphism (a)
- $F^\sharp \circ \alpha \sqsubseteq^\sharp \alpha \circ F^\natural$ (b)
- $\forall y \in \mathcal{D}^\sharp : F^\sharp(y) \sqsubseteq^\sharp y \Longrightarrow \exists x \in \mathcal{D}^\natural : \alpha(x) = y \wedge F^\natural(x) \sqsubseteq^\natural x$ (c)

then

$$\alpha(\mathsf{lfp}^{\sqsubseteq^\natural} F^\natural) = \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp$$

■

PROOF.

$$\begin{array}{lll}
\text{(d)} & F^\natural(x) \sqsubseteq^\natural x & \\
& \Longrightarrow \alpha \circ F^\natural(x) \sqsubseteq^\natural \alpha(x) & \text{since } \alpha \text{ is monotonic by (a)} \\
& \Longrightarrow F^\sharp \circ \alpha(x) \sqsubseteq^\natural \alpha(x) & \text{by (b)}
\end{array}$$

$$\text{(e)}\quad \{\alpha(x) \mid F^\natural(x) \sqsubseteq^\natural x\} = \{y \mid F^\sharp(y) \sqsubseteq^\sharp y\} \qquad \text{by (c) and (d)}$$

$$\begin{array}{lll}
\text{(f)} & \sqcap^\sharp\{\alpha(x) \mid F^\natural(x) \sqsubseteq^\natural x\} = \sqcap^\sharp\{y \mid F^\sharp(y) \sqsubseteq^\sharp y\} & \text{by (e)} \\
& \Longrightarrow \alpha(\sqcap^\natural\{x \mid F^\natural(x) \sqsubseteq^\natural x\}) = \sqcap^\sharp\{y \mid F^\sharp(y) \sqsubseteq^\sharp y\} & \text{by (a)} \\
& \Longrightarrow \alpha(\mathsf{lfp}^{\sqsubseteq^\natural} F^\natural) = \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp & \text{by Tarski's fixpt th.}
\end{array}$$

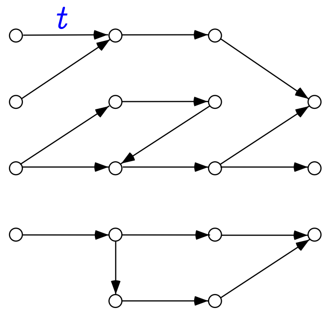
□

Example: application to reachability

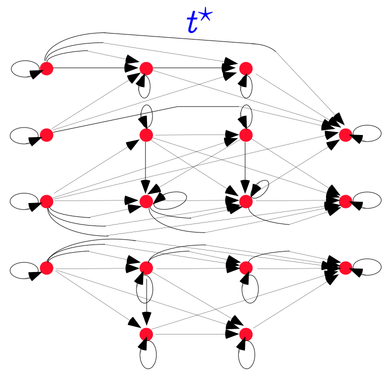
Transition systems

- $\langle S, t \rangle$ where:
  - S is a set of states/vertices/...
  - $t \in \wp(S \times S)$ is a transition relation/set of arcs/...

Example of transition system



The reflexive transitive closure of the example transition system



Reflexive closure of transition systems

Let r be a relation on x:

- $r^0 \stackrel{\text{def}}{=} 1_x$ powers
- $r^{n+1} \stackrel{\text{def}}{=} r^n \circ r \ (= r \circ r^n)$
- $r^\star \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} r^n$ reflexive transitive closure
- $r^+ \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N} \setminus \{0\}} r^n$ strict transitive closure

  so $r^\star = r^+ \cup 1_x$

Reflexive transitive closure in fixpoint form

$$t^* = \mathsf{lfp}^{\subseteq} \lambda X \cdot t^0 \cup X \circ t$$

PROOF.

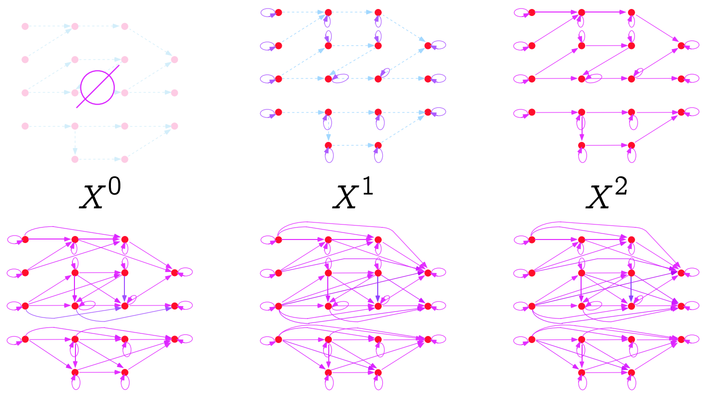
$$\begin{aligned}
X^0 &= \emptyset \\
X^1 &= t^0 \cup X^0 \circ t \ = \ t^0 \\
X^2 &= t^0 \cup X^1 \circ t \ = \ {}^0 \cup t^0 \circ t \ = \ t^0 \cup t^1 \\
\ldots & \quad \ldots \\
X^n &= \bigcup_{0 \le i < n} t^i \qquad \text{(induction hypothesis)}
\end{aligned}$$

$$\begin{aligned}
X^{n+1} &= t^0 \cup X^n \circ t \\
&= t^0 \cup \Big(\bigcup_{0 \le i < n} t^i\Big) \circ t \\
&= t^0 \cup \bigcup_{0 \le i < n} (t^i \circ t) \\
&= t^0 \cup \bigcup_{1 \le i+1 < n+1} (t^{i+1}) \\
&= t^0 \cup \Big(\bigcup_{1 \le j < n+1} t^j\Big) \circ t \\
&= \bigcup_{0 \le i < n+1} t^i
\end{aligned}$$

$$
\begin{aligned}
\ldots \quad & \ldots \\
X^\omega &= \bigcup_{n\geq 0} X^n \\
&= \bigcup_{n\geq 0} \bigcup_{0\leq i<n} t^i \\
&= \bigcup_{n\geq 0} t^n \\
&= t^*
\end{aligned}
\qquad
\begin{aligned}
X^{\omega+1} &= t^0 \cup X^\omega \circ t \\
&= t^0 \cup (\bigcup_{n\geq 0} t^n) \circ t \\
&= t^0 \cup \bigcup_{n\geq 0} (t^n \circ t) \\
&= t^0 \cup \bigcup_{n\geq 0} t^{n+1} \\
&= t^0 \cup \bigcup_{k\geq 1} t^k \\
&= \bigcup_{n\geq 0} t^n \\
&= t^*
\end{aligned}
$$

□

Examples of iterates

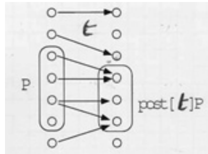


X^0 X^1 X^2

X^3 X^4 $X^5 = t^*$

Post-image

$$\text{post}[t]I = \{s' \mid \exists s \in I : \langle s, s'\rangle \in t\}$$



We have $\text{post}[\bigcup_{i\in\Delta} t^i]I = \bigcup_{i\in\Delta} \text{post}[t^i]I$ so $\alpha = \lambda t \cdot \text{post}[t]I$ is the lower adjoint of a Galois connection.

Postimage Galois connection

Given $I \in \wp(S)$,

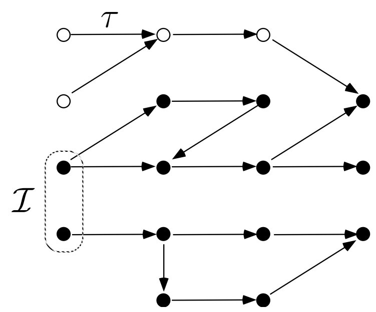
$$\langle \wp(S \times S), \subseteq\rangle \underset{\lambda t\cdot \text{post}[t]I}{\overset{\gamma}{\leftrightarrows}} \langle \wp(S), \subseteq\rangle$$

Proof.

$$
\begin{aligned}
& \text{post}[t]I \subseteq R \\
\Leftrightarrow\ & \{s' \mid \exists s \in I : \langle s, s'\rangle \in t\} \subseteq R \\
\Leftrightarrow\ & \forall s' \in S : (\exists s \in I : \langle s, s'\rangle \in t) \Rightarrow (s' \in R) \\
\Leftrightarrow\ & \forall s', s \in S : (s \in I \wedge \langle s, s'\rangle \in t) \Rightarrow (s' \in R) \\
\Leftrightarrow\ & \forall s', s \in S : \langle s, s'\rangle \in t \Rightarrow ((s \in I) \Rightarrow (s' \in R)) \\
\Leftrightarrow\ & t \subseteq \{\langle s, s'\rangle \mid (s \in I) \Rightarrow (s' \in R)\} \stackrel{\text{def}}{=} \gamma(R)
\end{aligned}
$$

□

Reachable states



$$\mathsf{post}[t^*]\mathcal{I}$$

Reachable states in fixpoint form

$$\begin{aligned}
&\mathsf{post}[t^*]I, \quad I \text{ given} \\
&= \alpha(t^*) \quad \text{where} \quad \alpha(t) = \mathsf{post}[t]I = \{s' \mid \exists s \in I : \langle s, s' \rangle \in t\} \\
&= \alpha(\mathsf{lfp}^{\subseteq}\, \lambda X \cdot t^0 \cup X \circ t) \\
&= \mathsf{lfp}^{\subseteq}\, \overline{F} \ ???
\end{aligned}$$

Discovering the abstract transformer by calculus

$$\begin{aligned}
&\alpha \circ (\lambda X \cdot t^0 \cup X \circ t) \\
&= \lambda X \cdot \alpha(t^0 \cup X \circ t) \\
&= \lambda X \cdot \alpha(t^0) \cup \alpha(X \circ t) \\
&= \lambda X \cdot \mathsf{post}[t^0]I \cup \mathsf{post}[X \circ t]I
\end{aligned}$$

$$\begin{aligned}
&\mathsf{post}[t^0]I \\
&= \{s' \mid \exists s \in I : \langle s, s' \rangle \in t^0\} \\
&= \{s' \mid \exists s \in I : \langle s, s' \rangle \in \{\langle s, s \rangle \mid s \in S\}\} \\
&= \{s' \mid \exists s \in I\} \\
&= I
\end{aligned}$$

$$
\begin{aligned}
& \text{post}[X \circ t]I \\
= \ & \{s' \mid \exists s \in I : \langle s, s' \rangle \in (X \circ t)\} \\
= \ & \{s' \mid \exists s \in I : \langle s, s' \rangle \in \{\langle s, s'' \rangle \mid \exists s' : \langle s, s" \rangle \in X \wedge \langle s', s'' \rangle \in t\}\} \\
= \ & \{s' \mid \exists s \in I : \exists s'' \in S : \langle s, s" \rangle \in X \wedge \langle s', s'' \rangle \in t\} \\
= \ & \{s' \mid \exists s'' \in S : (\exists s \in I : \langle s, s" \rangle \in X) \wedge \langle s', s'' \rangle \in t\} \\
= \ & \{s' \mid \exists s'' \in S : s'' \in \{s'' \mid \exists s \in I : \langle s, s" \rangle \in X\} \wedge \langle s', s'' \rangle \in t\} \\
= \ & \{s' \mid \exists s'' \in S : s'' \in \text{post}[X]I \wedge \langle s', s'' \rangle \in t\} \\
= \ & \text{post}[t](\text{post}[X]I) \\
= \ & \text{post}[t](\alpha(X))
\end{aligned}
$$

$$
\begin{aligned}
& \alpha \circ (\lambda X \cdot t^0 \cup X \circ t) \\
= \ & \ldots \\
= \ & \lambda X \cdot \text{post}[t^0]I \cup \text{post}[X \circ t]I \\
= \ & \lambda X \cdot I \cup \text{post}[t](\alpha(X)) \\
= \ & \lambda X \cdot \overline{F}(\alpha(X))
\end{aligned}
$$

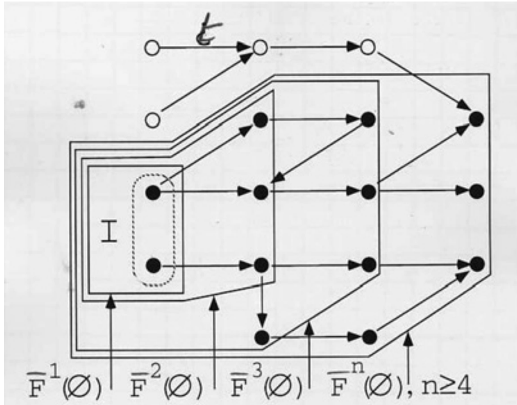
by defining:

$$\overline{F} = \lambda X \cdot I \cup \text{post}[t](X)$$

proving:

$$\text{post}[t^*](I) = \text{lfp}^{\subseteq} \lambda X \cdot I \cup \text{post}[t](X)$$

Example of iteration



THE END

My MIT web site is http://www.mit.edu/~cousot/

The course web site is http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/.