« Reachability and Postcondition Collecting Semantics »

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D'OPERATEURS MONOTONES SUR UN TREILLIS,
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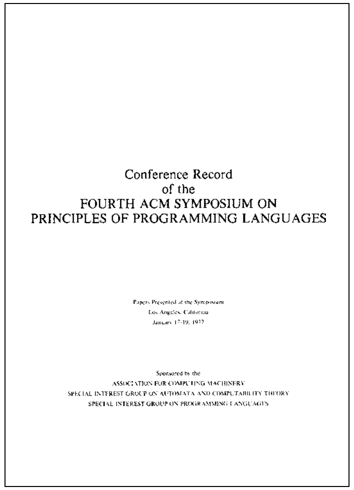

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Forward collecting semantics of arithmetic expressions

Definition of the forward collecting semantics of arithmetic expressions

Recall the *forward/bottom-up collecting semantics* of an arithmetic expression from lecture 8:

$$\text{Faexp} \in \text{Aexp} \mapsto \wp(\text{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathbb{I}_\Omega),$$

$$\text{Faexp}[\![A]\!]R \stackrel{\text{def}}{=} \{v \mid \exists \rho \in R : \rho \vdash A \Mapsto v\} . \qquad (1)$$

such that:

$$\text{Faexp}[\![A]\!]\left(\bigcup_{k \in S} R_k\right) = \bigcup_{k \in S} (\text{Faexp}[\![A]\!]R_k)$$

$$\text{Faexp}[\![A]\!]\emptyset = \emptyset .$$

Structural specification of the forward collecting semantics of arithmetic expressions

$$\mathrm{Faexp}[\![\mathtt{n}]\!]R = \{\underline{\mathtt{n}}\}$$ [1]

$$\mathrm{Faexp}[\![\mathtt{X}]\!]R = R(\mathtt{X})$$

where $R(\mathtt{X}) = \{\rho(\mathtt{X}) \mid \rho \in R\}$

$$\mathrm{Faexp}[\![?]\!]R = \mathbb{I}$$

$$\mathrm{Faexp}[\![\mathtt{u}\ A']\!]R = \underline{\mathtt{u}}^{\mathcal{C}}(\mathrm{Faexp}[\![A']\!]R)$$

where $\underline{\mathtt{u}}^{\mathcal{C}}(V) = \{\underline{\mathtt{u}}(v) \mid v \in V\}$

$$\mathrm{Faexp}[\![A_1\ \mathtt{b}\ A_2]\!]R = \underline{\mathtt{b}}^{\mathcal{C}}(\mathrm{Faexp}[\![A_1]\!], \mathrm{Faexp}[\![A_2]\!])R$$

where $\underline{\mathtt{b}}^{\mathcal{C}}(F_1, F_2)R = \{v_1\ \underline{\mathtt{b}}\ v_2 \mid \exists \rho \in R : v_1 \in F_1(\{\rho\}) \wedge v_2 \in F_2(\{\rho\})\}$

[1] For short, the case $\mathrm{Faexp}[\![A]\!]\emptyset = \emptyset$ is not recalled.

Forward collecting semantics of boolean expressions

Definition of the forward collecting semantics of boolean expressions

Recall the *collecting semantics* $\mathrm{Cbexp}[\![B]\!]R$ of a boolean expression B from lecture 8:

$$\mathrm{Cbexp} \in \mathrm{Bexp} \mapsto \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!]),$$

$$\mathrm{Cbexp}[\![B]\!]R \stackrel{\mathrm{def}}{=} \{\rho \in R \mid \rho \vdash B \Rightarrow \mathtt{tt}\}\ . \qquad (2)$$

such that:

$$\mathrm{Cbexp}[\![B]\!]\left(\bigcup_{k \in S} R_k\right) = \bigcup_{k \in S} (\mathrm{Cbexp}[\![B]\!]R_k)$$

$$\mathrm{Cbexp}[\![B]\!]\emptyset = \emptyset\ .$$

Structural specification of the forward collecting semantics of boolean expressions

$$\mathrm{Cbexp}[\![\mathtt{true}]\!]R = R$$

$$\mathrm{Cbexp}[\![\mathtt{false}]\!]R = \emptyset$$

$$\mathrm{Cbexp}[\![A_1\ \mathtt{c}\ A_2]\!] = \underline{\mathtt{c}}^{\mathcal{C}}(\mathrm{Faexp}[\![A_1]\!], \mathrm{Faexp}[\![A_2]\!])R$$

where $\underline{\mathtt{c}}^{\mathcal{C}}(F, G)R \stackrel{\mathrm{def}}{=} \{\rho \in R \mid \exists v_1 \in F(\{\rho\}) \cap \mathbb{I} : \exists v_2 \in G(\{\rho\}) \cap \mathbb{I} : v_1\ \underline{\mathtt{c}}\ v_2 = \mathtt{tt}\}$

$$\mathrm{Cbexp}[\![B_1\ \&\ B_2]\!]R = \mathrm{Cbexp}[\![B_1]\!]R \cap \mathrm{Cbexp}[\![B_2]\!]R$$

$$\mathrm{Cbexp}[\![B_1 \mid B_2]\!]R = (\mathrm{Cbexp}[\![B_1]\!]R \cap (\mathrm{Cbexp}[\![B_2]\!]R \cup \mathrm{Cbexp}[\![T(\neg B_2)]\!]R)) \cup (\mathrm{Cbexp}[\![B_2]\!]R \cap (\mathrm{Cbexp}[\![B_1]\!]R \cup \mathrm{Cbexp}[\![T(\neg B_1)]\!]R))$$

Big-step operational semantics of commands

Definition of the big-step operational semantics of commands

Recall the the big-step operational semantics of commands defined in course 8 as

$$\tau^\star[\![C]\!] \stackrel{\text{def}}{=} (\tau[\![C]\!])^\star.$$

where $\tau[\![C]\!]$ is the small-step operational semantics of the program components $C \in \text{Cmp}[\![P]\!]$ of program P.

Structural big-step operational semantics

$$\tau^\star[\![\texttt{skip}]\!] = 1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{skip}]\!] \tag{3}$$

where:

$$\tau[\![\texttt{skip}]\!] = \{\langle\langle \text{at}_P[\![\texttt{skip}]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{skip}]\!],\ \rho\rangle\rangle \mid \rho \in \text{Env}[\![P]\!]\}$$

$$\tau^\star[\![\texttt{X := } A]\!] = 1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{X := } A]\!] \tag{4}$$

where:

$$\tau[\![\texttt{X := } A]\!] = \{\langle\langle \text{at}_P[\![\texttt{X := } A]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{X := } A]\!],\ \rho[\texttt{X} := i]\rangle\rangle \mid i \in \mathbb{I} \wedge \rho \vdash A \Rrightarrow i\}$$

$$\tau^\star[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!] = (1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t) \ \cup\ (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f) \tag{5}$$

where:

$$\tau^B \stackrel{\text{def}}{=} \{\langle\langle \text{at}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle,\ \langle \text{at}_P[\![S_t]\!],\ \rho\rangle\rangle \mid \rho \vdash B \Rrightarrow \mathtt{tt}\}$$

$$\tau^{\bar{B}} \stackrel{\text{def}}{=} \{\langle\langle \text{at}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle,\ \langle \text{at}_P[\![S_f]\!],\ \rho\rangle\rangle \mid \rho \vdash T(\neg B) \Rrightarrow \mathtt{tt}\}$$

$$\tau^t \stackrel{\text{def}}{=} \{\langle\langle \text{after}_P[\![S_t]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle\rangle \mid \rho \in \text{Env}[\![P]\!]\}$$

$$\tau^f \stackrel{\text{def}}{=} \{\langle\langle \text{after}_P[\![S_f]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!],\ \rho\rangle\rangle \mid \rho \in \text{Env}[\![P]\!]\}$$

$$\tau^\star[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!] = \qquad (6)$$
$$((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ$$
$$(1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}})) \cup \tau[\![S]\!]^\star$$

where:

$$\tau^B \stackrel{\text{def}}{=} \{\langle\langle \text{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle,\ \langle \text{at}_P[\![S]\!],\ \rho\rangle\rangle \mid \rho \vdash B \mapsto \text{tt}\}$$
$$\tau^{\bar{B}} \stackrel{\text{def}}{=} \{\langle\langle \text{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle\rangle \mid \rho \vdash T(\neg B) \mapsto \text{tt}\}$$
$$\tau^R \stackrel{\text{def}}{=} \{\langle\langle \text{after}_P[\![S]\!],\ \rho\rangle,\ \langle \text{at}_P[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!],\ \rho\rangle\rangle \mid \rho \in \text{Env}[\![P]\!]\}$$

$$\tau^\star[\![C_1 \ ; \ \ldots \ ; \ C_n]\!] = \tau^\star[\![C_1]\!] \circ \ldots \circ \tau^\star[\![C_n]\!] \qquad (7)$$

$$\tau^\star[\![S \ ;;]\!] = \tau^\star[\![S]\!] \ . \qquad (8)$$

Postcondition collecting semantics of commands

Definition of the postcondition semantics of commands

The postcondition semantics $\text{Pcom}[\![C]\!]$ of a command $C \in \text{Com}$ (of a given program P) specifies the strongest postcondition $\text{Pcom}[\![C]\!]R$ satisfied by environments resulting from the execution of the command C starting in any of the environments satisfying the precondition R, if and when this execution terminates.

$$\text{Pcom} \in \text{Com} \mapsto \wp(\text{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\text{Env}[\![P]\!]) \qquad (9)$$
$$\text{Pcom}[\![C]\!]R \stackrel{\text{def}}{=} \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho\rangle,\ \langle \text{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \tau^\star[\![C]\!]\}$$

The postcondition semantics of a command can be understood, up to an interpretation, as a predicate transformer.

Property of the postcondition semantics of commands

The postcondition semantics of a command is a complete join morphism (denoted by $\stackrel{\sqcup}{\longmapsto}$) that is ($\mathcal{S}$ is an arbitrary set):

$$\text{Pcom}[\![C]\!]\left(\bigcup_{k \in \mathcal{S}} R_k\right) = \bigcup_{k \in \mathcal{S}} (\text{Pcom}[\![C]\!]R_k)$$

which implies monotony, continuity and strictness:

$$\text{Pcom}[\![C]\!]\emptyset = \emptyset \ .$$

Structural specification of the postcondition semantics of commands

$$
\begin{aligned}
\text{Pcom}[\![\texttt{skip}]\!]R &= R\\
\text{Pcom}[\![\texttt{X} := A]\!]R &= \{\rho[\texttt{X} := i] \mid \rho \in R \wedge i \in (\text{Faexp}[\![A]\!]\{\rho\}) \cap \mathbb{I}\}\\
\text{Pcom}[\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]R &= \\
&\quad \text{Pcom}[\![S_t]\!](\text{Cbexp}[\![B]\!]R) \cup \text{Pcom}[\![S_f]\!](\text{Cbexp}[\![T(\neg(B))]\!]R)\\
\text{Pcom}[\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!]R &= \\
&\quad \text{let } I = \text{lfp}_{\emptyset}^{\subseteq}\, \lambda X \cdot R \cup \text{Pcom}[\![S]\!](\text{Cbexp}[\![B]\!]X) \text{ in}\\
&\qquad \text{Cbexp}[\![T(\neg(B))]\!]I)\\
\text{Pcom}[\![C \;;\; S]\!]R &= (\text{Pcom}[\![S]\!] \circ \text{Pcom}[\![C]\!])R\\
\text{Pcom}[\![S \;;;]\!]R &= \text{Pcom}[\![S]\!]
\end{aligned}
$$

Postcondition semantics as an abstraction of the big-step operational semantics

If we let

$$
\begin{aligned}
\alpha[\![C]\!] &\in \wp(\Sigma[\![P]\!] \times \Sigma[\![P]\!]) \mapsto (\wp(\text{Env}[\![P]\!]) \mapsto \wp(\text{Env}[\![P]\!]))\\
\alpha[\![C]\!]X &= \lambda R \cdot \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in X\}
\end{aligned}
$$

then $\text{Pcom}[\![C]\!] = \alpha[\![C]\!](\tau^\star[\![C]\!])$ is an abstract interpretation of the big-step operational semantics $\tau^\star[\![C]\!]$.

Proof. – We observe that $\alpha[\![C]\!]$is a complete join morphism

$$
\begin{aligned}
&\alpha[\![C]\!](\bigcup_{i \in \Delta} X_i)\\
=\;& \lambda R \cdot \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in \bigcup_{i \in \Delta} X_i\}\\
=\;& \lambda R \cdot \bigcup_{i \in \Delta} \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in X_i\}\\
=\;& \dot{\bigcup_{i \in \Delta}} \lambda R \cdot \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in X_i\}\\
=\;& \dot{\bigcup_{i \in \Delta}} \alpha[\![C]\!] X_i
\end{aligned}
$$

so that it is a lower-adjoint of a Galois connection

$$
\langle \wp(\Sigma[\![P]\!] \times \Sigma[\![P]\!]),\, \subseteq\rangle \underset{\alpha[\![C]\!]}{\overset{\gamma[\![C]\!]}{\leftrightarrows}} \langle \wp(\text{Env}[\![P]\!]) \mapsto \wp(\text{Env}[\![P]\!]),\, \dot{\subseteq}\rangle
$$

for the pointwise extension $\dot{\subseteq}$ of $\subseteq$.

– We proceed by structural induction on the structure of programs.

$$
\begin{aligned}
-\;& \text{Pcom}[\![\texttt{skip}]\!]R\\
=\;& \alpha[\![\texttt{skip}]\!](\tau^\star[\![\texttt{skip}]\!])R\\
=\;& \alpha[\![\texttt{skip}]\!](1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{skip}]\!])R\\
=\;& \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{skip}]\!])\}\\
=\;& \{\rho' \mid \exists \rho \in R : \rho' = \rho\} \qquad \wr \text{at}_P[\![C]\!] \neq \text{after}_P[\![C]\!] \text{ and def. } \tau[\![\texttt{skip}]\!] \wr\\
=\;& R
\end{aligned}
$$

$$
\begin{aligned}
-\;& \text{Pcom}[\![\texttt{X} := A]\!]R\\
=\;& \alpha[\![\texttt{X} := A]\!](\tau^\star[\![\texttt{X} := A]\!])R\\
=\;& \alpha[\![\texttt{X} := A]\!](1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{X} := A]\!])R\\
=\;& \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau[\![\texttt{X} := A]\!])\}\\
=\;& \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho'\rangle\rangle \in \tau[\![\texttt{X} := A]\!]\} \qquad \wr \text{at}_P[\![C]\!] \neq \text{after}_P[\![C]\!] \wr
\end{aligned}
$$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \{\langle\langle \mathrm{at}_P[\![\mathtt{X} := A]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![\mathtt{X} := A]\!],\ \rho[\mathtt{X} := i]\rangle\rangle \mid i \in \mathbb{I} \wedge \rho \vdash A \Rightarrow i\}\}$

$\wr$def. $\tau[\![\mathtt{X} := A]\!]$$\int$

$= \{\rho[\mathtt{X} := i] \mid \rho \in R \wedge i \in \mathbb{I} \wedge \rho \vdash A \Rightarrow i\}$

$= \{\rho[\mathtt{X} := i] \mid \rho \in R \wedge i \in \{v \mid \rho \vdash A \Rightarrow v\} \cap \mathbb{I}\}$

$= \{\rho[\mathtt{X} := i] \mid \rho \in R \wedge i \in \{v \mid \exists \rho' \in \{\rho\} : \rho' \vdash A \Rightarrow v\} \cap \mathbb{I}\}$

$= \{\rho[\mathtt{X} := i] \mid \rho \in R \wedge i \in (\mathrm{Faexp}[\![A]\!](\{\rho\})) \cap \mathbb{I}\}$

— $\mathrm{Pcom}[\![C]\!]R$ where $C = \mathtt{if}\ B\ \mathtt{then}\ S_t\ \mathtt{else}\ S_f\ \mathtt{fi}$

$= \alpha[\![C]\!](\tau^\star[\![C]\!])R$

$= \alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)\ \cup$
$(1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f))R$

$=$ $\wr\alpha[\![C]\!]$ is a complete join morphism$\int$

$= \alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t))R\ \cup$
$\alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f))R$

We handle the case of the true alternative only, since the false alternativce can be handled in the same way.

$\alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t))R$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t))\}$ $\wr$by def. $\alpha[\![C]\!]$$\int$

$= \{\rho''' \mid \exists \rho \in R : \exists \rho', \rho'' \in \mathrm{Env}[\![P]\!] : \exists \ell', \ell'' \in \mathrm{in}_P[\![C]\!] : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \ell',\ \rho'\rangle\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau^B) \wedge \langle\langle \ell',\ \rho'\rangle,\ \langle \ell'',\ \rho''\rangle\rangle \in \tau^\star[\![S_t]\!] \wedge \langle\langle \ell'',\ \rho''\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'''\rangle\rangle \in (1_{\Sigma[\![P]\!]} \cup \tau^t)\}$ $\wr$def. composition $\circ$$\int$

$=$ $\wr$We observe that $\mathrm{at}_P[\![C]\!] = \ell'$ and $\mathrm{after}_P[\![C]\!] = \ell''$ is impossible since $\langle\langle \ell',\ \rho'\rangle,\ \langle \ell'',\ \rho''\rangle\rangle \in \tau^\star[\![S_t]\!]$ so $\ell', \ell'' \in \mathrm{in}_P[\![S_t]\!]$ in contradiction with $\{\mathrm{at}_P[\![C]\!], \mathrm{after}_P[\![C]\!]\} \cap (\mathrm{in}_P[\![S_t]\!] \cup \mathrm{in}_P[\![S_f]\!]) = \emptyset$$\int$

$\{\rho''' \mid \exists \rho \in R : \exists \rho', \rho'' \in \mathrm{Env}[\![P]\!] : \exists \ell', \ell'' \in \mathrm{in}_P[\![C]\!] : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \ell',\ \rho'\rangle\rangle \in \tau^B \wedge \langle\langle \ell',\ \rho'\rangle,\ \langle \ell'',\ \rho''\rangle\rangle \in \tau^\star[\![S_t]\!] \wedge \langle\langle \ell'',\ \rho''\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'''\rangle\rangle \in \tau^t\}$

$=$ $\wr$By def. of τ^B, we have $\mathrm{at}_P[\![S_t]\!] = \ell'$ and by def. of τ^t, we have $\mathrm{after}_P[\![S_t]\!] = \ell''$$\int$

$\{\rho''' \mid \exists \rho \in R : \exists \rho', \rho'' \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{at}_P[\![S_t]\!],\ \rho'\rangle\rangle \in \tau^B \wedge \langle\langle \mathrm{at}_P[\![S_t]\!],\ \rho'\rangle,\ \langle \mathrm{after}_P[\![S_t]\!],\ \rho''\rangle\rangle \in \tau^\star[\![S_t]\!] \wedge \langle\langle \mathrm{after}_P[\![S_t]\!],\ \rho''\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'''\rangle\rangle \in \tau^t\}$

$=$ $\wr$By def. τ^t so that $\rho'' = \rho'''$$\int$

$\{\rho''' \mid \exists \rho \in R : \exists \rho' \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{at}_P[\![S_t]\!],\ \rho'\rangle\rangle \in \tau^B \wedge \langle\langle \mathrm{at}_P[\![S_t]\!],\ \rho'\rangle,\ \langle \mathrm{after}_P[\![S_t]\!],\ \rho'''\rangle\rangle \in \tau^\star[\![S_t]\!]\}$

$=$ $\wr$By def. τ^B so that $\rho' = \rho$ and $\rho' \vdash B \Rightarrow \mathtt{tt}$$\int$

$\{\rho''' \mid \exists \rho \in R : \rho \vdash B \Rightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P[\![S_t]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![S_t]\!],\ \rho'''\rangle\rangle \in \tau^\star[\![S_t]\!]\}$

$= \{\rho' \mid \exists \rho \in \{\rho'' \in R \mid \rho'' \vdash B \Rightarrow \mathtt{tt}\} : \langle\langle \mathrm{at}_P[\![S_t]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![S_t]\!],\ \rho'\rangle\rangle \in \tau^\star[\![S_t]\!]\}$

$\wr$By def. (2) of $\mathrm{Cbexp}[\![B]\!]R$$\int$

$= \{\rho' \mid \exists \rho \in \mathrm{Cbexp}[\![B]\!]R : \langle\langle \mathrm{at}_P[\![S_t]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![S_t]\!],\ \rho'\rangle\rangle \in \tau^\star[\![S_t]\!]\}$

$= \mathrm{Pcom}[\![S_t]\!](\mathrm{Cbexp}[\![B]\!]R)$

The false alternative is similar with S_f for S_t and $\neg(B)$ for B.

— $\mathrm{Pcom}[\![C]\!]R$ where $C = \mathtt{while}\ B\ \mathtt{do}\ S\ \mathtt{od}$

$= \alpha[\![C]\!](\tau^\star[\![C]\!])R$

$= \alpha[\![C]\!](((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}})) \cup \tau[\![S]\!]^\star)R$

$=$ $\wr\alpha[\![C]\!]$ is a complete join morphism$\int$

$\alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}}))R \cup \alpha[\![C]\!](\tau[\![S]\!]^\star)R$

$=$ $\wr$def. $\alpha[\![C]\!]$$\int$

$\alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}}))R \cup \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \tau[\![S]\!]^\star\}$

$=$ $\wr$For the second term, we have $\langle\langle \mathrm{at}_P[\![C]\!],\ \rho\rangle,\ \langle \mathrm{after}_P[\![C]\!],\ \rho'\rangle\rangle \in \tau[\![S]\!]^\star$ which implies $\{\mathrm{at}_P[\![C]\!], \mathrm{after}_P[\![C]\!]\} \subseteq \mathrm{in}_P[\![S]\!]$ and so $\{\mathrm{at}_P[\![C]\!], \mathrm{after}_P[\![C]\!]\} \cap \mathrm{in}_P[\![S]\!] = \emptyset$ implies that this term is $\emptyset$$\int$

$\alpha[\![C]\!]((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}}))R$

$=$ $\wr$def. $\alpha[\![C]\!]$$\int$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^\star[\![S]\!] \circ \tau^R) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^B \circ \tau^\star[\![S]\!] \cup \tau^{\bar{B}}))\}$

$=$ $\wr$ $\mathrm{at}_P[\![C]\!] \notin \mathrm{in}_P[\![S]\!]$ so $\langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \ell'', \rho''\rangle\rangle \notin \tau^\star[\![S]\!] \circ \tau^R$ and $\mathrm{after}_P[\![C]\!] \notin \mathrm{in}_P[\![S]\!]$ so $\langle\langle \ell, \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \notin \tau^B \circ \tau^\star[\![S]\!]$ $\wr$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in ((1_{\Sigma[\![P]\!]}) \circ (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}))\}$

$=$ $\wr$ $1_{\Sigma[\![P]\!]}$ neutral element of $\circ$ $\wr$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in ((\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}))\}$

$=$ $\wr$ $\langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \notin 1_{\Sigma[\![P]\!]}$ since $\mathrm{at}_P[\![C]\!] \neq \mathrm{after}_P[\![C]\!]$ so the term $((\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}))$ cannot reduce to $1_{\Sigma[\![P]\!]}$. Moreover $\langle\langle \ell, \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \not{f} \circ \tau^R$ so the term $((\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ (1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}))$ cannot either reduce to $(\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^+$ $\wr$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in ((\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ \tau^{\bar{B}})\}$

$=$ $\wr$ def. composition $\circ$ $\wr$

$\{\rho' \mid \exists \rho \in R : \exists \ell'' \in \mathrm{in}_P[\![C]\!] : \exists \rho'' \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \ell'', \rho''\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \langle\langle \ell'', \rho''\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in \tau^{\bar{B}}\}$

$=$ $\wr$ def. $\tau^{\bar{B}}$ so that $\ell'' = \mathrm{at}_P[\![C]\!]$ and $\rho'' = \rho'$ $\wr$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \langle\langle \mathrm{at}_P[\![C]\!], \rho'\rangle, \langle \mathrm{after}_P[\![C]\!], \rho'\rangle\rangle \in \tau^{\bar{B}}\}$

$=$ $\wr$ def. $\tau^{\bar{B}}$ $\wr$

$\{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \rho' \vdash T(\neg B) \Rightarrow \mathrm{tt}\}$

$=$ let $I \stackrel{\mathrm{def}}{=} \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star\}$ in
$\{\rho' \in I \mid \rho' \vdash T(\neg B) \Rightarrow \mathrm{tt}\}$

$=$ let $I \stackrel{\mathrm{def}}{=} \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star\}$ in
$\mathrm{Cbexp}[\![T(\neg B)]\!]I$

$=$ let $I \stackrel{\mathrm{def}}{=} \alpha'((\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star)$ in
$\mathrm{Cbexp}[\![T(\neg B)]\!]I$

by defining (for a given program P, command C and set of environments R):

$$\alpha'(t) \stackrel{\mathrm{def}}{=} \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in t\}$$

We have

$$\alpha'(\bigcup_{i \in \Delta} t_i) = \bigcup_{i \in \Delta} \alpha'(t_i)$$

whence a Galois connection

$$\langle \wp(\Sigma[\![P]\!] \times \Sigma[\![P]\!]), \subseteq\rangle \underset{\alpha'}{\overset{\gamma'}{\leftrightarrows}} \langle \wp(\mathbb{R}), \subseteq\rangle$$

such that

$$I = \alpha'(t^\star) \quad \text{where} \quad t = (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)$$
$$= \alpha'(\mathbf{lfp}\, \lambda X \cdot 1_{\Sigma[\![P]\!]} \cup X \circ t)$$
$$= \mathbf{lfp}\, F'$$

where $\alpha'(1_{\Sigma[\![P]\!]} \cup X \circ t) = F'(\alpha'(X))$ that is $\alpha'(1_{\Sigma[\![P]\!]}) \cup \alpha'(X \circ t) = F'(\alpha'(X))$ since α' is a complete join morphism.

— $\alpha'(X \circ t)$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in X \circ t\}$ $\wr$ def. α' $\wr$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in X \circ \tau^B \circ \tau^\star[\![S]\!] \circ \tau^R\}$ $\wr$ def. t $\wr$

$= \{\rho' \mid \exists \rho \in R : \exists \ell_1, \ell_2, \ell_3 \in \mathrm{in}_P[\![C]\!] : \exists \rho_1, \rho_2, \rho_3 \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \ell_1, \rho_1\rangle\rangle \in X \wedge \langle\langle \ell_1, \rho_1\rangle, \langle \ell_2, \rho_2\rangle\rangle \in \tau^B \wedge \langle\langle \ell_2, \rho_2\rangle, \langle \ell_3, \rho_3\rangle\rangle \in \tau^\star[\![S]\!] \wedge \langle\langle \ell_3, \rho_3\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in \tau^R\}$ $\wr$ def. $\circ$ $\wr$

$=$ $\wr$ by def. τ^B, we have $\ell_1 = \mathrm{at}_P[\![C]\!]$, $\ell_2 = \mathrm{at}_P[\![S]\!]$, $\rho_2 = \rho_1$ and $\rho_1 \vdash B \Rightarrow \mathrm{tt}$ $\wr$

$\{\rho' \mid \exists \rho \in R : \exists \ell_3 \in \mathrm{in}_P[\![C]\!] : \exists \rho_1, \rho_3 \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho_1\rangle\rangle \in X \wedge \rho_1 \vdash B \Rightarrow \mathrm{tt} \wedge \langle\langle \mathrm{at}_P[\![S]\!], \rho_1\rangle, \langle \ell_3, \rho_3\rangle\rangle \in \tau^\star[\![S]\!] \wedge \langle\langle \ell_3, \rho_3\rangle, \langle \mathrm{at}_P[\![C]\!], \rho'\rangle\rangle \in \tau^R\}$

$=$ $\wr$ by def. τ^R, we have $\ell_3 = \mathrm{after}_P[\![S]\!]$ and $\rho_3 = \rho'$ $\wr$

$\{\rho' \mid \exists \rho \in R : \exists \rho_1 \in \mathrm{Env}[\![P]\!] : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho_1\rangle\rangle \in X \wedge \rho_1 \vdash B \Rightarrow \mathrm{tt} \wedge \langle\langle \mathrm{at}_P[\![S]\!], \rho_1\rangle, \langle \mathrm{after}_P[\![S]\!], \rho'\rangle\rangle \in \tau^\star[\![S]\!]\}$

$= \{\rho' \mid \exists \rho_1 \in \{\rho_1 \in \mathrm{Env}[\![P]\!] \mid \exists \rho \in R : \langle\langle \mathrm{at}_P[\![C]\!], \rho\rangle, \langle \mathrm{at}_P[\![C]\!], \rho_1\rangle\rangle \in X\} : \rho_1 \vdash B \Rightarrow \mathrm{tt} \wedge \langle\langle \mathrm{at}_P[\![S]\!], \rho_1\rangle, \langle \mathrm{after}_P[\![S]\!], \rho'\rangle\rangle \in \tau^\star[\![S]\!]\}$

$= \{\rho' \mid \exists \rho_1 \in \alpha'(X) : \rho_1 \vdash B \Rrightarrow \mathrm{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$
$\wr$by def. $\alpha'$$\wr$
$= \{\rho' \mid \exists \rho_1 \in \{\rho_1 \in \alpha'(X) \mid \rho_1 \vdash B \Rrightarrow \mathrm{tt}\} \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$
$= \{\rho' \mid \exists \rho_1 \in \mathrm{Cbexp}\llbracket B\rrbracket(\alpha'(X)) \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$
$= \mathrm{Pcom}\llbracket S\rrbracket(\mathrm{Cbexp}\llbracket B\rrbracket(\alpha'(X)))$

— $\alpha'(1_{\Sigma\llbracket P\rrbracket})$
$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle\rangle \in 1_{\Sigma\llbracket P\rrbracket}\}$ $\wr$def. $\alpha'$$\wr$
$= \{\rho' \mid \exists \rho \in R : \rho = \rho'\}$
$= R$

So $F'(X) = R \cup \mathrm{Pcom}\llbracket S\rrbracket(\mathrm{Cbexp}\llbracket B\rrbracket(X))$ and $I = \mathsf{lfp}^{\subseteq}_{\emptyset}\ F'$.

— $\mathrm{Pcom}\llbracket C\ ;\ S\rrbracket R$
$= \alpha\llbracket C\ ;\ S\rrbracket(\tau^\star\llbracket C\ ;\ S\rrbracket)R$
$= \alpha\llbracket C\ ;\ S\rrbracket(\tau^\star\llbracket C\rrbracket \circ \tau^\star\llbracket S\rrbracket)R$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\ ;\ S\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\ ;\ S\rrbracket,\ \rho'\rangle\rangle \in (\tau^\star\llbracket C\rrbracket \circ \tau^\star\llbracket S\rrbracket)\}$
$=$ $\wr \mathrm{at}_P\llbracket C\ ;\ S\rrbracket = \mathrm{at}_P\llbracket C\rrbracket$ and $\mathrm{after}_P\llbracket C\ ;\ S\rrbracket = \mathrm{after}_P\llbracket S\rrbracket\wr$
$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in (\tau^\star\llbracket C\rrbracket \circ \tau^\star\llbracket S\rrbracket)\}$
$= \{\rho' \mid \exists \rho \in R : \exists \ell'' \in \mathrm{in}_P\llbracket C\ ;\ S\rrbracket : \exists \rho'' \in \mathbb{R} : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \ell'',\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket \wedge \langle\langle \ell'',\ \rho''\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket)\}$
$=$ $\wr\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \ell'',\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket$ so $\ell'' \in \mathrm{in}_P\llbracket C\rrbracket$ and $\langle\langle \ell'',\ \rho''\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket$ so $\ell'' \in \mathrm{in}_P\llbracket S\rrbracket$. Hence $\ell'' \in \mathrm{in}_P\llbracket C\rrbracket \cap \mathrm{in}_P\llbracket S\rrbracket$ proving $\ell'' = \mathrm{after}_P\llbracket C\rrbracket = \mathrm{at}_P\llbracket S\rrbracket\wr$
$\{\rho' \mid \exists \rho \in R : \exists \rho'' \in \mathbb{R} : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho''\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket)\}$
$= \{\rho' \mid \exists \rho'' \in \mathbb{R} : \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho''\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket)\}$
$= \{\rho' \mid \exists \rho'' \in \{\rho'' \mid \mathbb{R} : \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket\} : \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho''\rangle,\ \langle \mathrm{after}_P\llbracket S\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket)\}$
$= \mathrm{Pcom}\llbracket S\rrbracket(\{\rho'' \mid \mathbb{R} : \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho''\rangle\rangle \in \tau^\star\llbracket C\rrbracket\})$
$= \mathrm{Pcom}\llbracket S\rrbracket(\mathrm{Pcom}\llbracket C\rrbracket(R))$

$= (\mathrm{Pcom}\llbracket S\rrbracket \circ \mathrm{Pcom}\llbracket C\rrbracket)(R)$

— $\mathrm{Pcom}\llbracket S\ ;;\rrbracket R$
$= \alpha\llbracket S\ ;;\rrbracket(\tau^\star\llbracket S\ ;;\rrbracket)R$
$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\ ;;\rrbracket\}$
$= \{\rho' \mid \exists \rho \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho'\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$
$= \mathrm{Pcom}\llbracket S\rrbracket R$

$\square$

Reminder of the small-step operational semantics of commands

Small-step operational semantics of commands and programs

Identity $C = \texttt{skip}$ $(\mathsf{at}_P[\![C]\!] = \ell \neq \ell' = \mathsf{after}_P[\![C]\!])$

$$\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{skip}]\!]\!\!\Longrightarrow \langle \ell',\ \rho\rangle \qquad (19)$$

Assignment $C = \texttt{X := } A$ $(\mathsf{at}_P[\![C]\!] = \ell \neq \ell' = \mathsf{after}_P[\![C]\!])$

$$\frac{\rho \vdash A \Rrightarrow i}{\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{X := } A]\!]\!\!\Longrightarrow \langle \ell',\ \rho[\texttt{X} := i]\rangle},\ i \in \mathbb{I} \qquad (20)$$

Conditional $C = \texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}$ $(\mathsf{at}_P[\![C]\!] = \ell \neq \ell' = \mathsf{after}_P[\![C]\!])$

$$\frac{\rho \vdash B \Rrightarrow \mathtt{tt}}{\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \mathsf{at}_P[\![S_t]\!],\ \rho\rangle} \qquad (21)$$

$$\mathsf{at}_P[\![S_t]\!] \neq \mathsf{at}_P[\![C]\!]$$

$$\frac{\rho \vdash T(\neg B) \Rrightarrow \mathtt{tt}}{\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \mathsf{at}_P[\![S_f]\!],\ \rho\rangle} \qquad (22)$$

$$\mathsf{at}_P[\![S_f]\!] \neq \mathsf{at}_P[\![C]\!]$$

$$\frac{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![S_t]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle}{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle} \qquad (23)$$

$$\ell_2 \neq \mathsf{at}_P[\![C]\!]$$

$$\frac{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![S_f]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle}{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle} \qquad (24)$$

$$\ell_2 \neq \mathsf{at}_P[\![C]\!]$$

$$\langle \mathsf{after}_P[\![S_t]\!],\ \rho\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \ell',\ \rho\rangle \qquad (25)$$

$$\ell' = \mathsf{after}_P[\![C]\!] \neq \mathsf{at}_P[\![C]\!]$$

$$\langle \mathsf{after}_P[\![S_f]\!],\ \rho\rangle \longmapsto\!\![\![\texttt{if } B \texttt{ then } S_t \texttt{ else } S_f \texttt{ fi}]\!]\!\!\Longrightarrow \langle \ell',\ \rho\rangle \qquad (26)$$

$$\ell' = \mathsf{after}_P[\![C]\!] \neq \mathsf{at}_P[\![C]\!]$$

Iteration $C = \texttt{while } B \texttt{ do } S \texttt{ od}$ $(\mathsf{at}_P[\![C]\!] = \ell,$ $\mathsf{after}_P[\![C]\!] = \ell'$ and $\ell_1, \ell_2 \in \mathsf{in}_P[\![S]\!])$

$$\frac{\rho \vdash T(\neg B) \Rrightarrow \mathtt{tt}}{\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!]\!\!\Longrightarrow \langle \ell',\ \rho\rangle} \qquad (27)$$

$$\frac{\rho \vdash B \Rrightarrow \mathtt{tt}}{\langle \ell,\ \rho\rangle \longmapsto\!\![\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!]\!\!\Longrightarrow \langle \mathsf{at}_P[\![S]\!],\ \rho\rangle} \qquad (28)$$

$$\frac{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![S]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle}{\langle \ell_1,\ \rho_1\rangle \longmapsto\!\![\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!]\!\!\Longrightarrow \langle \ell_2,\ \rho_2\rangle} \qquad (29)$$

$$\langle \mathsf{after}_P[\![S]\!],\ \rho\rangle \longmapsto\!\![\![\texttt{while } B \texttt{ do } S \texttt{ od}]\!]\!\!\Longrightarrow \langle \ell,\ \rho\rangle \qquad (30)$$

Sequence C_1 ; ... ; C_n, $n > 0$ ($i \in [1,n]$:$\ell_i, \ell_{i+1} \in \mathrm{in}_P[\![C_i]\!]$)

$$\frac{\langle \ell_i,\ \rho_i\rangle \Longmapsto\!\![\![C_i]\!]\!\!\Longrightarrow \langle \ell_{i+1},\ \rho_{i+1}\rangle}{\langle \ell_i,\ \rho_i\rangle \Longmapsto\!\![\![C_1\ ;\ \ldots\ ;\ C_n]\!]\!\!\Longrightarrow \langle \ell_{i+1},\ \rho_{i+1}\rangle} \tag{31}$$

$\mathrm{after}_P[\![C_i]\!] = \mathrm{at}_P[\![C_{i+1}]\!]$

Program $P = S$;;

$$\frac{\langle \ell,\ \rho\rangle \Longmapsto\!\![\![S]\!]\!\!\Longrightarrow \rho'}{\langle \ell,\ \rho\rangle \Longmapsto\!\![\![S\ ;;]\!]\!\!\Longrightarrow \langle \ell',\ \rho'\rangle} \tag{32}$$

$\ell' = \mathrm{after}_P[\![P]\!] = \mathrm{after}_P[\![S]\!] \neq \mathrm{at}_P[\![S]\!] = \mathrm{at}_P[\![P]\!]$

Transition system of a program

The transition system of a program $P = S$;; is

$$\langle \Sigma[\![P]\!],\ \tau[\![P]\!]\rangle$$

where $\Sigma[\![P]\!]$ is the set of program states and $\tau[\![C]\!]$, $C \in \mathrm{Cmp}[\![P]\!]$ is the transition relation for component C of program P, defined by

$$\Sigma[\![P]\!] \stackrel{\text{def}}{=} \mathrm{in}_P[\![P]\!] \times \mathrm{Env}[\![P]\!] \tag{33}$$

$$\tau[\![C]\!] \stackrel{\text{def}}{=} \{\langle\langle \ell,\ \rho\rangle,\ \langle \ell',\ \rho'\rangle\rangle \mid \langle \ell,\ \rho\rangle \Longmapsto\!\![\![C]\!]\!\!\Longrightarrow \langle \ell',\ \rho'\rangle\} \tag{34}$$

Forward reachability collecting semantics of commands

Definition of the forward reachability collecting semantics of commands

The forward reachability collecting semantics $\mathrm{Rcom}[\![C]\!]R$ of a command $C \in \mathrm{Com}$ (of a given program P) specifies the set of reachable sattes during any execution of C starting at its starting point in any of the enviornments satisfying the precondition R.

$$\mathrm{Rcom}\ \in\ \mathrm{Com} \mapsto \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} (\mathrm{in}_P[\![C]\!] \mapsto \wp(\mathrm{Env}[\![P]\!]))$$

$$\mathrm{Rcom}[\![C]\!]R\ell \stackrel{\text{def}}{=} \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star[\![C]\!]\}$$

Property of the forward reachability collecting semantics of commands

The forward reachability collecting semantics of a command is a complete join morphism (denoted by $\stackrel{\sqcup}{\longmapsto}$) that is ($\mathcal{S}$ is an arbitrary set):

$$\mathrm{Rcom}[\![C]\!]\Big(\bigcup_{k \in \mathcal{S}} R_k\Big) = \bigcup_{k \in \mathcal{S}} (\mathrm{Rcom}[\![C]\!]R_k)$$

which implies monotony, continuity and strictness:

$$\mathrm{Rcom}[\![C]\!]\emptyset = \emptyset \ .$$

Postcondition semantics as an abstraction of the forward reachability collecting semantics

We have

$$\mathrm{Pcom}[\![C]\!]R = \mathrm{Rcom}[\![C]\!]R(\mathrm{after}_P[\![C]\!]) \qquad (35)$$

which is an abstraction $\alpha[\![C]\!](\mathrm{Rcom}[\![C]\!]R)$ of a function at a point, by defining $\alpha[\![C]\!](f) = f(\mathrm{after}_P[\![C]\!])$ such that

$$\langle \mathrm{Com} \mapsto (\wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!])),\ \dot{\subseteq}\rangle \xrightleftharpoons[\alpha[\![C]\!]]{\gamma[\![C]\!]} \langle \wp(\mathrm{Env}[\![P]\!]) \stackrel{\sqcup}{\longmapsto} \wp(\mathrm{Env}[\![P]\!]),\ \subseteq\rangle$$

So we could have first designed $\mathrm{Rcom}[\![C]\!]$ from $\tau^\star[\![C]\!]$ and then $\mathrm{Pcom}[\![C]\!]$ from $\mathrm{Rcom}[\![C]\!]$ [2]

[2] but for pedagogical reasons, we first designed $\mathrm{Pcom}[\![C]\!]$ directly from $\tau^\star[\![C]\!]$ thinking that this would be more simple. Moreoever, $\mathrm{Rcom}[\![C]\!]$ contains fixpoint terms already computed for $\mathrm{Pcom}[\![C]\!]$ which makes the presentation more modular.

Structural definition of the forward reachability collecting semantics of commands

$\mathrm{Rcom}[\![\mathtt{skip}]\!]R\ell = R$

$\mathrm{Rcom}[\![\mathtt{X} := A]\!]R\ell = \mathsf{match}\ \ell\ \mathsf{with}$
$\mid \mathsf{at}_P[\![\mathtt{X} := A]\!] \to R$
$\mid \mathsf{after}_P[\![\mathtt{X} := A]\!] \to \{\rho[\mathtt{X} := i] \mid \rho \in R \wedge i \in (\mathrm{Faexp}[\![A]\!]\{\rho\}) \cap \mathbb{I}\}$

$\mathrm{Rcom}[\![C]\!]R\ell$ where $C = \mathtt{if}\ B\ \mathtt{then}\ S_t\ \mathtt{else}\ S_f\ \mathtt{fi} =$
$\quad \mathsf{match}\ \ell\ \mathsf{with}$
$\qquad \mid \mathsf{at}_P[\![C]\!] \to R$
$\qquad \mid \mathsf{in}_P[\![S_t]\!] \to \mathrm{Rcom}[\![S_t]\!](\mathrm{Cbexp}[\![B]\!]R)\ell$
$\qquad \mid \mathsf{in}_P[\![S_f]\!] \to \mathrm{Rcom}[\![S_f]\!](\mathrm{Cbexp}[\![T(\neg(B))]\!]R)\ell$
$\qquad \mid \mathsf{after}_P[\![C]\!] \to \mathrm{Rcom}[\![S_t]\!](\mathrm{Cbexp}[\![B]\!]R)(\mathsf{after}_P[\![S_t]\!])$
$\qquad\qquad \cup \mathrm{Rcom}[\![S_f]\!](\mathrm{Cbexp}[\![T(\neg(B))]\!]R)(\mathsf{after}_P[\![S_f]\!])$

$\mathrm{Rcom}[\![C]\!]R\ell$ where $C = \mathtt{while}\ B\ \mathtt{do}\ S\ \mathtt{od} =$
$\quad \mathsf{let}\ I = \mathsf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot R \cup \mathrm{Rcom}[\![S]\!](\mathrm{Cbexp}[\![B]\!]X)(\mathsf{after}_P[\![S]\!])\ \mathsf{in}$
$\qquad \mathsf{match}\ \ell\ \mathsf{with}$
$\qquad \mid \mathsf{at}_P[\![C]\!] \to I$
$\qquad \mid \mathsf{in}_P[\![S]\!] \to \mathrm{Rcom}[\![S]\!](\mathrm{Cbexp}[\![B]\!]I)(\ell)$
$\qquad \mid \mathsf{after}_P[\![C]\!] \to \mathrm{Cbexp}[\![T(\neg(B))]\!]R)I$

$\mathrm{Rcom}[\![C\ ;\ S]\!]R\ell = \mathsf{match}\ \ell\ \mathsf{with}$
$\quad \mid \mathsf{in}_P[\![C]\!] \to \mathrm{Rcom}[\![C]\!]R\ell$
$\quad \mid \mathsf{in}_P[\![S]\!] \to \mathrm{Rcom}[\![S]\!](\mathrm{Rcom}[\![C]\!]R(\mathsf{after}_P[\![C]\!]))\ell$

$\mathrm{Rcom}[\![S\ ;;]\!]R\ell = \mathrm{Rcom}[\![S]\!]R\ell$

Proof. By structural induction on the abstract syntax of programs. We first distinguish the special case of $\text{Rcom}[\![C]\!]R\ell$ where $\ell = \text{at}_P[\![C]\!]$ and $C \neq$ `while` B `do` S `od`

– $\text{Rcom}[\![C]\!]R(\text{at}_P[\![C]\!])$ when $C \neq$ `while` B `do` S `od`

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \text{at}_P[\![C]\!],\ \rho\rangle\rangle \in \tau^\star[\![C]\!]\}$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \text{at}_P[\![C]\!],\ \rho\rangle\rangle \in \bigcup_{n\in\mathbb{N}} \tau[\![C]\!]^n\}$

$=$ ⟅From the definition of $\langle \ell,\ \rho\rangle \stackrel{}{\longmapsto\!\![\![C]\!]\!\!\Longrightarrow} \langle \ell',\ \rho'\rangle$ and that of $\langle\langle \ell,\ \rho\rangle,\ \langle \ell',\ \rho'\rangle\rangle \in \tau[\![C]\!]$, if follows that $\ell' \neq \text{at}_P[\![C]\!]$ whence whenever $n > 0$, $\langle\langle \ell,\ \rho\rangle,\ \langle \ell',\ \rho'\rangle\rangle \in \tau[\![C]\!]^n$ implies $\ell' \neq \text{at}_P[\![C]\!]$⟆

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \text{at}_P[\![C]\!],\ \rho\rangle\rangle \in \tau[\![C]\!]^0\}$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \text{at}_P[\![C]\!],\ \rho\rangle\rangle \in 1_{\Sigma[\![P]\!]}\}$

$= \{\rho \mid \exists \rho' \in R : \rho' = \rho\}$

$= R$

We now define $\text{Rcom}[\![C]\!]R\ell$ assuming, but for the case of while loops, that $\ell \in \text{in}_P[\![C]\!] \setminus \{\text{at}_P[\![C]\!]\}$. We proceed by structural induction on C.

– $\text{Rcom}[\![\texttt{skip}]\!]R\ell$

⟅The only case is $\ell = \text{after}_P[\![\texttt{skip}]\!]$⟆

$\text{Rcom}[\![\texttt{skip}]\!]R(\text{after}_P[\![\texttt{skip}]\!]$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![\texttt{skip}]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{skip}]\!],\ \rho'\rangle\rangle \in \tau^\star[\![\texttt{skip}]\!]\}$

$= \{\rho' \mid \exists \rho \in R : \langle \text{at}_P[\![\texttt{skip}]\!],\ \rho\rangle \longmapsto\!\![\![\texttt{skip}]\!]\!\!\Longrightarrow \langle \text{after}_P[\![\texttt{skip}]\!],\ \rho'\rangle\}$

$= \{\rho' \mid \exists \rho \in R : \rho = \rho'\}$

$= R$

– $\text{Rcom}[\![\texttt{X} := A]\!]R\ell$

⟅The only case is $\ell = \text{after}_P[\![\texttt{X} := A]\!]$⟆

$\text{Rcom}[\![\texttt{X} := A]\!]R(\text{after}_P[\![\texttt{X} := A]\!]$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![\texttt{X} := A]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{X} := A]\!],\ \rho'\rangle\rangle \in \tau^\star[\![\texttt{X} := A]\!]\}$

$= \{\rho' \mid \exists \rho \in R : \langle\langle \text{at}_P[\![\texttt{X} := A]\!],\ \rho\rangle,\ \langle \text{after}_P[\![\texttt{X} := A]\!],\ \rho'\rangle\rangle \in \tau[\![\texttt{X} := A]\!]\}$

$= \{\rho' \mid \exists \rho \in R : \langle \text{at}_P[\![\texttt{X} := A]\!],\ \rho\rangle \longmapsto\!\![\![\texttt{X} := A]\!]\!\!\Longrightarrow \langle \text{after}_P[\![\texttt{X} := A]\!],\ \rho'\rangle\}$

$= \{\rho[\texttt{X} := i] \mid \rho \in R \wedge i \in \mathbb{I} \wedge \rho \vdash A \Mapsto i\}$

$= \{\rho[\texttt{X} := i] \mid i \in \{v \mid \exists \rho' \in \{\rho\} : \rho' \vdash A \Mapsto i\} \cap \mathbb{I}\}$

$= \{\rho[\texttt{X} := i] \mid \rho \in R \wedge i \in (\text{Faexp}[\![A]\!]\{\rho\}) \cap \mathbb{I}\}$

– $\text{Rcom}[\![C]\!]R\ell$ where $C =$ `if` B `then` S_t `else` S_f `fi`

⟅The only cases are $\ell = \text{after}_P[\![C]\!]$, $\ell \in \text{in}_P[\![S_t]\!]$ and $\ell \in \text{in}_P[\![S_f]\!]$. The two last cases are similar and we handle only one.⟆

– $\text{Rcom}[\![C]\!]R\ell$ where $\ell \in \text{in}_P[\![S_t]\!]$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star[\![C]\!] \wedge \ell \in \text{in}_P[\![S_t]\!]\}$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)) \cup ((1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f))) \wedge \ell \in \text{in}_P[\![S_t]\!]\}$

$=$ ⟅If $\langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^f))$ then

- either $\langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau^\star[\![S_f]\!] \circ \tau^f)$ in which case $\ell = \text{after}_P[\![C]\!]$;
- or $\langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^{\bar{B}}) \circ \tau[\![S_f]\!]^+)$ in which case $\ell \in \text{in}_P[\![S_f]\!]$;
- or $\langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^{\bar{B}}$ in which case $\ell = \text{at}_P[\![S_f]\!]$;
- or $\langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in 1_{\Sigma[\![P]\!]}$ in which case $\ell = \text{at}_P[\![C]\!]$.

In all cases, this is in contradiction with $\ell \in \text{in}_P[\![S_t]\!]$ so this case is impossible.⟆

$\{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((1_{\Sigma[\![P]\!]} \cup \tau^B) \circ \tau^\star[\![S_t]\!] \circ (1_{\Sigma[\![P]\!]} \cup \tau^t)) \wedge \ell \in \text{in}_P[\![S_t]\!]\}$

$= \quad \wr$If $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in (\tau^\star\llbracket S_t\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t))$ then

- either $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in (\tau\llbracket S_t\rrbracket^+ \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t))$, in which case $\mathrm{at}_P\llbracket C\rrbracket \in \mathrm{in}_P\llbracket S_t\rrbracket$, which is impossible
- or $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^t$, in which case $\mathrm{at}_P\llbracket C\rrbracket = \mathrm{after}_P\llbracket S_t\rrbracket$, which is excluded
- or $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in 1_{\Sigma\llbracket P\rrbracket}$ and so $\mathrm{at}_P\llbracket C\rrbracket = \ell$ is contradiction with $\mathrm{at}_P\llbracket C\rrbracket \notin \mathrm{in}_P\llbracket S_t\rrbracket$

$\wr$

$\{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S_t\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t)) \wedge \ell \in \mathrm{in}_P\llbracket S_t\rrbracket\}$

$= \quad \wr$def. composition $\circ\wr$

$\{\rho \mid \exists\rho' \in R : \exists\rho_1,\rho_2 \in \mathbb{R} : \exists\ell_1,\ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \ell \in \mathrm{in}_P\llbracket S_t\rrbracket \wedge \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell_1,\ \rho_1\rangle\rangle \in \tau^B \wedge \langle\langle\ell_1,\ \rho_1\rangle,\ \langle\ell_2,\ \rho_2\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \wedge \langle\langle\ell_2,\ \rho_2\rangle,\ \langle\ell,\ \rho\rangle\rangle \in (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t))\}$

$= \quad \wr$def. $\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho\rangle\rangle \mid \rho' \vdash B \Rrightarrow \mathtt{tt}\}\wr$

$\{\rho \mid \exists\rho' \in R : \exists\rho_2 \in \mathbb{R} : \exists\ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \ell \in \mathrm{in}_P\llbracket S_t\rrbracket \wedge \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell_2,\ \rho_2\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \wedge \langle\langle\ell_2,\ \rho_2\rangle,\ \langle\ell,\ \rho\rangle\rangle \in (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t))\}$

$= \quad \wr$y def. $\tau^t \stackrel{\text{def}}{=} \{\langle\langle \mathrm{after}_P\llbracket S_t\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \mid \rho \in \mathrm{Env}\llbracket P\rrbracket\}$, the case $\langle\langle\ell_2,\ \rho_2\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^t$ implies $\ell = \mathrm{after}_P\llbracket C\rrbracket$, in contradiction with $\ell \in \mathrm{in}_P\llbracket S_t\rrbracket\wr$

$\{\rho \mid \exists\rho' \in R : \exists\rho_2 \in \mathbb{R} : \exists\ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \ell \in \mathrm{in}_P\llbracket S_t\rrbracket \wedge \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell_2,\ \rho_2\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \wedge \langle\langle\ell_2,\ \rho_2\rangle,\ \langle\ell,\ \rho\rangle\rangle \in 1_{\Sigma\llbracket P\rrbracket}\}$

$= \{\rho \mid \exists\rho' \in R : \ell \in \mathrm{in}_P\llbracket S_t\rrbracket \wedge \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$

$= \quad \wr\langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket$ implies $\ell \in \mathrm{in}_P\llbracket S_t\rrbracket\wr$

$\{\rho \mid \exists\rho' \in R : \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$

$= \{\rho \mid \exists\rho' \in \{\rho' \in R \mid \rho' \vdash B \Rrightarrow \mathtt{tt}\} : \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$

$= \{\rho \mid \exists\rho' \in \mathrm{Cbexp}\llbracket B\rrbracket R : \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$ $\wr$def. $\mathrm{Cbexp}\llbracket B\rrbracket R\wr$

$= \mathrm{Rcom}\llbracket S_t\rrbracket(\mathrm{Cbexp}\llbracket B\rrbracket R)\ell$ $\wr$def. $\mathrm{Rcom}\llbracket S_t\rrbracket\wr$

– $\mathrm{Rcom}\llbracket C\rrbracket R\ell$ where $\ell \in \mathrm{in}_P\llbracket S_f\rrbracket$

$= \mathrm{Rcom}\llbracket S_f\rrbracket(\mathrm{Cbexp}\llbracket T(\neg(B))\rrbracket R)\ell$, in the same way

– $\mathrm{Rcom}\llbracket C\rrbracket R(\mathrm{after}_P\llbracket C\rrbracket)$ where $C = \mathtt{if}\ B\ \mathtt{then}\ S_t\ \mathtt{else}\ S_f\ \mathtt{fi}$

$= \{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket C\rrbracket\}$

$= \{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in ((1_{\Sigma\llbracket P\rrbracket} \cup \tau^B) \circ \tau^\star\llbracket S_t\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t) \cup (1_{\Sigma\llbracket P\rrbracket} \cup \tau^{\bar{B}}) \circ \tau^\star\llbracket S_f\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^f))\}$

$= \quad \wr$The case $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t)$ would imply $\mathrm{at}_P\llbracket C\rrbracket \in (\mathrm{in}_P\llbracket S_t\rrbracket \cup \{\mathrm{after}_P\llbracket S_t\rrbracket\})$ by def. $\tau^\star\llbracket S_t\rrbracket$ and τ^t, or $\mathrm{at}_P\llbracket C\rrbracket = \mathrm{after}_P\llbracket C\rrbracket$ which is impossible. The same way, $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket S_f\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^f)$ is impossible.$\wr$

$\{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S_t\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^t) \cup \tau^{\bar{B}} \circ \tau^\star\llbracket S_f\rrbracket \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^f))\}$

$= \quad \wr$If $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^B \circ \tau^\star\llbracket S_t\rrbracket$ then $\mathrm{after}_P\llbracket C\rrbracket \in \mathrm{in}_P\llbracket S_t\rrbracket$ by def. $\tau\llbracket S_t\rrbracket^+$ or $\mathrm{after}_P\llbracket C\rrbracket = \mathrm{at}_P\llbracket S_t\rrbracket$, which is impossible. The same way, $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^{\bar{B}} \circ \tau^\star\llbracket S_f\rrbracket$ is impossible.$\wr$

$\{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S_t\rrbracket \circ \tau^t \cup \tau^{\bar{B}} \circ \tau^\star\llbracket S_f\rrbracket \circ \tau^f)\}$

$= \{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^B \circ \tau^\star\llbracket S_t\rrbracket \circ \tau^t\}$
$\cup \{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^{\bar{B}} \circ \tau^\star\llbracket S_f\rrbracket \circ \tau^f\}$

Since both cases are identical, we handle the first one.

$\{\rho \mid \exists\rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^B \circ \tau^\star\llbracket S_t\rrbracket \circ \tau^t\}$

$= \{\rho \mid \exists\rho' \in R : \exists\rho_1,\rho_2 \in \mathrm{Env}\llbracket P\rrbracket : \exists : \ell_1,\ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle\ell_1,\ \rho_1\rangle\rangle \in \tau^B \wedge \langle\langle\ell_1,\ \rho_1\rangle,\ \langle\ell_2,\ \rho_2\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \wedge \langle\langle\ell_2,\ \rho_2\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^t\}$ $\wr$by def. $\circ\wr$

$= \quad \wr$By def. $\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho\rangle\rangle \mid \rho \vdash B \Rrightarrow \mathtt{tt}\}$, $\rho_1 = \rho'$, $\ell_1 = \mathrm{at}_P\llbracket S_t\rrbracket$ and $\rho' \vdash B \Rrightarrow \mathtt{tt}\wr$

$\{\rho \mid \exists\rho' \in R : \exists\rho_2 \in \mathrm{Env}\llbracket P\rrbracket : \exists : \ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle\ell_2,\ \rho_2\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket \wedge \langle\langle\ell_2,\ \rho_2\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^t\}$

$= \quad \wr$By def. $\tau^t \stackrel{\text{def}}{=} \{\langle\langle \mathrm{after}_P\llbracket S_t\rrbracket,\ \rho\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \mid \rho \in \mathrm{Env}\llbracket P\rrbracket\}$, we have $\ell_2 = \mathrm{after}_P\llbracket S_t\rrbracket$ and $\rho_2 = \rho\wr$

$\{\rho \mid \exists\rho' \in R : \rho' \vdash B \Rrightarrow \mathtt{tt} \wedge \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket S_t\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$

$$= \{\rho \mid \exists \rho' \in \{\rho' \in\in R : \rho' \vdash B \Rrightarrow tt\} : \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket S_t\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$$
$$= \{\rho \mid \exists \rho' \in \mathrm{Cbexp}\llbracket B\rrbracket R : \langle\langle \mathrm{at}_P\llbracket S_t\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket S_t\rrbracket,\ \rho\rangle\rangle \in \tau^\star\llbracket S_t\rrbracket\}$$
$$= \mathrm{Rcom}\llbracket S_t\rrbracket(\mathrm{Cbexp}\llbracket B\rrbracket R)(\mathrm{after}_P\llbracket S_t\rrbracket)$$

The same way, we have

$$\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{after}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in \tau^{\bar{B}} \circ \tau^\star\llbracket S_f\rrbracket \circ \tau^f\}$$
$$= \mathrm{Rcom}\llbracket S_f\rrbracket(\mathrm{Cbexp}\llbracket T(\neg(B))\rrbracket R)(\mathrm{after}_P\llbracket S_f\rrbracket)$$

— $\mathrm{Rcom}\llbracket C\rrbracket R\ell$ where $C = \texttt{while } B \texttt{ do } S \texttt{ od}$

$$= \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star\llbracket C\rrbracket\}$$
$$= \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (((1_{\Sigma\llbracket P\rrbracket} \cup \tau^\star\llbracket S\rrbracket \circ \tau^R) \circ (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^B \circ \tau^\star\llbracket S\rrbracket \cup \tau^{\bar{B}})) \cup \tau\llbracket S\rrbracket^\star)\}$$
$= \quad \wr$The case $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau\llbracket S\rrbracket^\star$ implies $\mathrm{at}_P\llbracket C\rrbracket \in \mathrm{in}_P\llbracket S\rrbracket$, which is impossible$\wr$

$$\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((1_{\Sigma\llbracket P\rrbracket} \cup \tau^\star\llbracket S\rrbracket \circ \tau^R) \circ (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^B \circ \tau^\star\llbracket S\rrbracket \cup \tau^{\bar{B}}))\}$$
$= \quad \wr$The same way, $\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((\tau^\star\llbracket S\rrbracket \circ \tau^R) \circ (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^B \circ \tau^\star\llbracket S\rrbracket \cup \tau^{\bar{B}}))$ implies either $\mathrm{at}_P\llbracket C\rrbracket \in \mathrm{in}_P\llbracket S\rrbracket$ by def. $\tau\llbracket S\rrbracket$ or $\mathrm{at}_P\llbracket C\rrbracket = \mathrm{after}_P\llbracket S\rrbracket$, by def. τ^R and both cases are impossible$\wr$

$$\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in ((\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ (1_{\Sigma\llbracket P\rrbracket} \cup \tau^B \circ \tau^\star\llbracket S\rrbracket \cup \tau^{\bar{B}}))\}$$
$$= \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star\}$$
$$\cup\ \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ \tau^B \circ \tau^\star\llbracket S\rrbracket\}$$
$$\cup\ \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ \tau^{\bar{B}}\}$$

We study all three cases separately.

- $\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star\}$

$= \quad \wr$
– in case the reflexive transitive closure is the identity, then obviously $\ell = \mathrm{at}_P\llbracket C\rrbracket$;
– in case the reflexive transitive closure is not the identity, then, by definition of $\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho\rangle\rangle \mid \rho \vdash B \Rrightarrow tt\}$, $\ell = \mathrm{at}_P\llbracket C\rrbracket$.

So when $\ell \neq \mathrm{at}_P\llbracket C\rrbracket$, the expression is empty.$\wr$

match ℓ with
$\mid \mathrm{at}_P\llbracket C\rrbracket \to \underbrace{\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star\}}_{I}$
$\mid\ \_ \to \emptyset$

$= \quad \wr$as already computed on pages 27 to 29$\wr$

match ℓ with
$\mid \mathrm{at}_P\llbracket C\rrbracket \to \mathbf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot R \cup \mathrm{Pcom}\llbracket S\rrbracket(\mathrm{Cbexp}\llbracket B\rrbracket(X))$
$\mid\ \_ \to \emptyset$

= match ℓ with
$\mid \mathrm{at}_P\llbracket C\rrbracket \to \underbrace{\mathbf{lfp}_{\emptyset}^{\subseteq}\ \lambda X \cdot R \cup \mathrm{Rcom}\llbracket S\rrbracket\ \mathrm{Cbexp}\llbracket B\rrbracket(X)(\mathrm{after}_P\llbracket S\rrbracket)}_{I}$
$\mid\ \_ \to \emptyset$ $\quad\quad \wr$by (35)$\wr$

- $\{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \circ \tau^B \circ \tau^\star\llbracket S\rrbracket\}$

$= \quad \wr$def. composition $\circ\wr$

$$\{\rho \mid \exists \rho' \in R : \exists \rho_1, \rho_2 \in \mathbb{R} : \exists \ell_1, \ell_2 \in \mathrm{in}_P\llbracket C\rrbracket : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \ell_1,\ \rho_1\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \wedge \langle\langle \ell_1,\ \rho_1\rangle,\ \langle \ell_2,\ \rho_2\rangle\rangle \in \tau^B \wedge \langle\langle \ell_2,\ \rho_2\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$$

$= \quad \wr$By def. $\tau^B \stackrel{\text{def}}{=} \{\langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle,\ \langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho\rangle\rangle \mid \rho \vdash B \Rrightarrow tt\}$, we have $\ell_1 = \mathrm{at}_P\llbracket C\rrbracket$, $\ell_2 = \mathrm{at}_P\llbracket S\rrbracket$, $\rho_1 = \rho_2$ and $\rho_1 \vdash B \Rrightarrow tt\wr$

$$\{\rho \mid \exists \rho' \in R : \exists \rho_1 \in \mathbb{R} : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho_1\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star \wedge \rho_1 \vdash B \Rrightarrow tt \wedge \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$$

$= \quad \wr$By def. $I = \{\rho \mid \exists \rho' \in R : \langle\langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho'\rangle,\ \langle \mathrm{at}_P\llbracket C\rrbracket,\ \rho\rangle\rangle \in (\tau^B \circ \tau^\star\llbracket S\rrbracket \circ \tau^R)^\star\}\wr$

$$\{\rho \mid \exists \rho_1 \in \{\rho_1 \in I \mid \rho_1 \vdash B \Rrightarrow tt\} : \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$$
$$= \{\rho \mid \exists \rho_1 \in \mathrm{Cbexp}\llbracket B\rrbracket I : \langle\langle \mathrm{at}_P\llbracket S\rrbracket,\ \rho_1\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star\llbracket S\rrbracket\}$$

$= \; (\!|\, \ell \in \text{in}_P[\![S]\!] \;?\; \text{Rcom}[\![S]\!](\text{Cbexp}[\![B]\!]I)\ell \;\grave{\imath}\; \emptyset \,|\!)$

- $\{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \circ \tau^{\bar{B}}\}$

$=$ $\wr$def. composition $\circ\int$

$\{\rho \mid \exists \rho' \in R : \exists \rho_1 \in \mathbb{R} : \exists \ell_1 \in \text{in}_P[\![C]\!] : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell_1,\, \rho_1\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \langle\langle \ell_1,\, \rho_1\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^{\bar{B}}\}$

$=$ $\wr$def. $\tau^{\bar{B}} \stackrel{\text{def}}{=} \{\langle\langle \text{at}_P[\![C]\!],\, \rho\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho\rangle\rangle \mid \rho \vdash T(\neg B) \Rightarrow \text{tt}\}$ so that $\ell_1 = \text{at}_P[\![C]\!]$, $\rho_1 = \rho$, $\ell = \text{after}_P[\![C]\!]$ and $\rho \vdash T(\neg B) \Rightarrow \text{tt}\int$

$\{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \text{at}_P[\![C]\!],\, \rho\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \rho \vdash T(\neg B) \Rightarrow \text{tt} \wedge \ell = \text{after}_P[\![C]\!]\}$

$=$ match ℓ with
| $\text{after}_P[\![C]\!] \to \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \text{at}_P[\![C]\!],\, \rho\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star \wedge \rho \vdash T(\neg B) \Rightarrow \text{tt}\}$
| $\_ \to \emptyset$

$=$ match ℓ with
| $\text{after}_P[\![C]\!] \to \{\rho \in \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \text{at}_P[\![C]\!],\, \rho\rangle\rangle \in (\tau^B \circ \tau^\star[\![S]\!] \circ \tau^R)^\star\} \mid \rho \vdash T(\neg B) \Rightarrow \text{tt}\}$
| $\_ \to \emptyset$

$=$ match ℓ with
| $\text{after}_P[\![C]\!] \to \{\rho \in I \mid \rho \vdash T(\neg B) \Rightarrow \text{tt}\}$
| $\_ \to \emptyset$

$=$ match ℓ with
| $\text{after}_P[\![C]\!] \to \text{Cbexp}[\![T(\neg(B))]\!]I$
| $\_ \to \emptyset$

- Grouping all cases of the $\cup$ together, we get:

$\text{Rcom}[\![C]\!]R\ell$ where $C = \texttt{while}\ B\ \texttt{do}\ S\ \texttt{od}$

$=$ let $I = \textsf{lfp}^{\subseteq}_{\emptyset}\; \lambda X \cdot E \cup \text{Rcom}[\![S]\!](\text{Cbexp}[\![B]\!]X)(\text{after}_P[\![S]\!])$ in
match ℓ with
| $\text{at}_P[\![C]\!] \to I$
| $\text{in}_P[\![S]\!] \to \text{Rcom}[\![S]\!](\text{Cbexp}[\![B]\!]I)(\ell)$
| $\text{after}_P[\![C]\!] \to \text{Cbexp}[\![T(\neg(B))]\!]R)I$

– $\text{Rcom}[\![C\ ;\ S]\!]R\ell$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C\ ;\ S]\!]\}$

$= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C]\!] \circ \tau^\star[\![S]\!]\}$

$= \{\rho \mid \exists \rho' \in R : \exists \rho_1 \in \mathbb{R} : \exists \ell_1 \in \text{in}_P[\![C]\!] : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell_1,\, \rho_1\rangle\rangle \in \tau^\star[\![C]\!] \wedge \langle\langle \ell_1,\, \rho_1\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![S]\!]\}$

$=$ $\wr$Distinguishing the cases when $\ell\,\text{in}_P[\![C]\!]$ (when $\tau^\star[\![S]\!]$ is the identity) or $\ell\,\text{in}_P[\![S]\!]$ and noting in this second case that $\ell_1 = \text{after}_P[\![C]\!] = \text{at}_P[\![S]\!] \in \text{in}_P[\![C]\!] \cap \text{in}_P[\![S]\!]\int$

match ℓ with
| $\text{in}_P[\![C]\!] \to \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C]\!]\}$
| $\text{in}_P[\![S]\!] \to \{\rho \mid \exists \rho' \in R : \exists \rho_1 \in \mathbb{R} : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho_1\rangle\rangle \in \tau^\star[\![C]\!] \wedge \langle\langle \text{at}_P[\![S]\!],\, \rho_1\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![S]\!]\}$
| $\_ \to \emptyset$

$=$ $\wr$The last case is indeed impossible since $\ell \in \text{in}_P[\![C\ ;\ S]\!] = \text{in}_P[\![C]\!] \cup \text{in}_P[\![S]\!]\int$

match ℓ with
| $\text{in}_P[\![C]\!] \to \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C]\!]\}$
| $\text{in}_P[\![S]\!] \to \{\rho \mid \exists \rho_1 \in \{\rho_1 \in \mathbb{R} \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \text{after}_P[\![C]\!],\, \rho_1\rangle\rangle \in \tau^\star[\![C]\!]\} : \langle\langle \text{at}_P[\![S]\!],\, \rho_1\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![S]\!]\}$

$=$ $\wr$def. $\text{Rcom}[\![C]\!]R\ell \stackrel{\text{def}}{=} \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C]\!]\}\int$

match ℓ with
| $\text{in}_P[\![C]\!] \to \text{Rcom}[\![C]\!]R\ell$
| $\text{in}_P[\![S]\!] \to \{\rho \mid \exists \rho_1 \in \text{Rcom}[\![C]\!]R(\text{after}_P[\![C]\!]) : \langle\langle \text{at}_P[\![S]\!],\, \rho_1\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![S]\!]\}$

$=$ $\wr$def. $\text{Rcom}[\![C]\!]R\ell \stackrel{\text{def}}{=} \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![C]\!],\, \rho'\rangle,\, \langle \ell,\, \rho\rangle\rangle \in \tau^\star[\![C]\!]\}\int$

match ℓ with
| $\text{in}_P[\![C]\!] \rightarrow \text{Rcom}[\![C]\!]R\ell$
| $\text{in}_P[\![S]\!] \rightarrow \text{Rcom}[\![S]\!](\text{Rcom}[\![C]\!]R(\text{after}_P[\![C]\!]))\ell$

$$
\begin{aligned}
&- \text{Rcom}[\![S\ ;;]\!]R\ell \\
&= \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![S\ ;;]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star[\![S\ ;;]\!]\} \\
&= \quad \wr \text{at}_P[\![S\ ;;]\!] = \text{at}_P[\![S]\!] \text{ and } \tau^\star[\![S\ ;;]\!] = \tau^\star[\![S]\!] \wr \\
&\quad \{\rho \mid \exists \rho' \in R : \langle\langle \text{at}_P[\![S]\!],\ \rho'\rangle,\ \langle \ell,\ \rho\rangle\rangle \in \tau^\star[\![S]\!]\} \\
&= \text{Rcom}[\![S]\!]R\ell
\end{aligned}
$$

□

Implementation of the forward reachability collecting

Implementability of the forward reachability collecting semantics

- The foward reachability collecting semantics collects all values and environments by simulation of all possible executions.
- The main problem is random input (?) because of the very large number of possible values (e.g. between `min_int=-1.073.741.834` and `max_int=-1.073.741.833`) which makes exhaustive simulation impossible in practice
- We will limit the possible values of the random input (?) to 3 instead of 2.147.483.648

- Apart from this restriction, the implementation is faithful to the definition of the forward reachability collecting semantics in that it simulates all possible executions of the program (for 3 of the possible random values at each random expression computation)
- Despite this restriction, the forward reachability collecting semantics is rapidly subject to a combinatorial explosion of the states and therefore is useless in practice.

Example: trace of the fixpoint computation with two nested loops

I — no iteration within the loops

```
% cat ../Examples/example12-1.sil
% example12-1.sil %
n := 1;
x := 1;
while (x < n) do
        x := x + 1;
        a := ?;
        y := 1;
        while (y < n) do
                y := y + 1;
                b := ?
        od
od;;
```



```
% ./a.out ../Examples/example12-1.sil
...

iterate 0 = { }
iterate 0 = { }
fixpoint = { }
iterate 1 = { [ n = 1; x = 1; a = _O_(i); y = _O_(i); b = _O_(i); ] }
iterate 0 = { }
fixpoint = { }
fixpoint = { [ n = 1; x = 1; a = _O_(i); y = _O_(i); b = _O_(i); ] }
{ [ n = 1; x = 1; a = _O_(i); y = _O_(i); b = _O_(i); ] }
% ^Dexit

Script done on Mon Apr  4 12:37:52 2005
```


II — 10 iterations within the loops

Page 1:

```
Script started on Mon Apr  4 12:45:41 2005
% make trace

ocamlyacc parser.mly
ocamllex lexer.mll
62 states, 3001 transitions, table size 12376 bytes
ocamlc symbol_Table.mli symbol_Table.ml variables.mli variables.ml
abstract_Syntax.ml concrete_To_Abstract_Syntax.mli
concrete_To_Abstract_Syntax.ml labels.mli labels.ml parser.mli
parser.ml lexer.ml program_To_Abstract_Syntax.mli
program_To_Abstract_Syntax.ml pretty_Print.mli pretty_Print.ml
values.mli values.ml cvalues.mli cvalues.ml env.mli env.ml cenv.mli
cenv.ml caexp.mli caexp.ml cbexp.mli cbexp.ml fixpoint.mli fixpoint.ml
ccom.mli ccom.ml main.ml
fixpoint tracing mode
```



```
% cat ../Examples/example12-10.sil

% example12-10.sil %
n := 10;
x := 1;
while (x < n) do
x := x + 1;
a := ?;
y := 1;
while (y < n) do
y := y + 1;
b := ?
od
od;;
```

```
% ./a.out ../Examples/example12-10.sil
...
iterate 0 = { }
iterate 0 = { }
fixpoint = { }
iterate 1 = { [ n = 10; x = 1; a = _O_(i); y = _O_(i); b = _O_(i); ] }
iterate 0 = { }
iterate 1 = { [ n = 10; x = 2; a = -819618235; y = 1; b = _O_(i); ]
              [ n = 10; x = 2; a = -361540549; y = 1; b = _O_(i); ]
              [ n = 10; x = 2; a = 625724514; y = 1; b = _O_(i); ] }
iterate 2 = { [ n = 10; x = 2; a = -819618235; y = 1; b = _O_(i); ]
              [ n = 10; x = 2; a = -819618235; y = 2; b = -819618235; ]
              [ n = 10; x = 2; a = -819618235; y = 2; b = -361540549; ]
              [ n = 10; x = 2; a = -819618235; y = 2; b = 625724514; ]
              [ n = 10; x = 2; a = -361540549; y = 1; b = _O_(i); ]
              [ n = 10; x = 2; a = -361540549; y = 2; b = -819618235; ]
```


Page 471 (showing the invariant on loop exit):

```
{ [ n = 10; x = 10; a = -819618235; y = 10; b = -819618235; ]
  [ n = 10; x = 10; a = -819618235; y = 10; b = -361540549; ]
  [ n = 10; x = 10; a = -819618235; y = 10; b = 625724514; ]
  [ n = 10; x = 10; a = -361540549; y = 10; b = -819618235; ]
  [ n = 10; x = 10; a = -361540549; y = 10; b = -361540549; ]
  [ n = 10; x = 10; a = -361540549; y = 10; b = 625724514; ]
  [ n = 10; x = 10; a = 625724514; y = 10; b = -819618235; ]
  [ n = 10; x = 10; a = 625724514; y = 10; b = -361540549; ]
  [ n = 10; x = 10; a = 625724514; y = 10; b = 625724514; ] }
```

An exhaustive simulation would involve 2.147.483.648 $\times$ 2.147.483.648 cases, instead of the above $3 \times 3 = 9$. Note that choosing *different* random values at each random assignment would ultimately cover all machine integers whence explode combinatorially.

Totally ordered types in OCaml

From `http://caml.inria.fr/pub/docs/manual-ocaml/libref/Set.OrderedType.html`:

`module type OrderedType = sig .. end`

Input signature of the functor `Set.Make`

`type t`

The type of the set elements.

`val compare : t -> t -> int`

A total ordering function over the set elements. This is a two-argument function `f` such that `f e1 e2` is zero if the elements `e1` and `e2` are equal, `f e1 e2` is strictly negative if `e1` is smaller than `e2`, and `f e1 e2` is strictly positive if `e1` is greater than `e2`.

Implementation: sets in OCaml

From `http://caml.inria.fr/pub/docs/manual-ocaml/libref/Set.html`:

`module Set: sig .. end`

Sets over ordered types implemented using balanced binary trees, no side-effects.

`module type S = sig .. end`

Output signature of the functor `Set.Make`

`module Make: functor (Ord : OrderedType) -> S with type elt = Ord.t`

Functor building an implementation of the set structure given a totally ordered type.

Example: ordered set of machine integers

```
(* ordered set of machine integers *)
include Set.Make
   (struct
      type t = machine_int
      (* order on values *)
      let compare v1 v2 = match v1, v2 with
         | (ERROR_NAT INITIALIZATION), (ERROR_NAT INITIALIZATION) -> 0
         | (ERROR_NAT INITIALIZATION), (ERROR_NAT ARITHMETIC) -> -1
         | (ERROR_NAT ARITHMETIC), (ERROR_NAT INITIALIZATION) -> 1
         | (ERROR_NAT ARITHMETIC), (ERROR_NAT ARITHMETIC) -> 0
         | (ERROR_NAT e), (NAT i) -> -1
         | (NAT i), (ERROR_NAT e) -> 1
         | (NAT i), (NAT j) -> if (i<j) then -1 else
                                  if (i=j) then 0 else 1
   end)
```


Signatures

From `http://caml.inria.fr/pub/docs/manual-ocaml/libref/type_Set.html`:

```
module Make :
  functor (Ord : OrderedType) ->
    sig
      type elt = Ord.t
      type t
      val add : elt -> t -> t
      val elements : t -> elt list
      val empty : t
      val equal : t -> t -> bool
      val filter : (elt -> bool) -> t -> t
      val fold : (elt -> 'a -> 'a) -> t -> 'a -> 'a
      val for_all : (elt -> bool) -> t -> bool
      val inter : t -> t -> t
      val is_empty : t -> bool
      val iter : (elt -> unit) -> t -> unit
      val singleton : elt -> t
      val subset : t -> t -> bool
      val union : t -> t -> t
    end
```


Specification of set operations

- `add` $e\ s \stackrel{\text{def}}{=} s \cup \{e\}$
- `elements` $\emptyset \stackrel{\text{def}}{=}$ `[]`
- `elements` $\{a_1, \ldots, a_n\} \stackrel{\text{def}}{=} [a_1; \ldots; a_n]$
- `empty` $\stackrel{\text{def}}{=} \emptyset$
- `equal` $s_1\ s_2 \stackrel{\text{def}}{=} (s_1 = s_2\ ?\ \mathrm{tt}\ ¿\ \mathrm{ff})$
- `filter` $p\ s \stackrel{\text{def}}{=} \{x \in s \mid p(x)\}$
- `fold` $f\ \emptyset\ b \stackrel{\text{def}}{=} b$
- `fold` $f\ \{a_1, \ldots, a_n\}\ b \stackrel{\text{def}}{=} f\ a_1\ (f a_2\ (\ldots (f\ a_n\ b) \ldots))$

- `inter` $s_1\ s_2 \stackrel{\text{def}}{=} s_1 \cap s_2$
- `is_empty` $s \stackrel{\text{def}}{=} (s = \emptyset\ ?\ \mathrm{tt}\ ¿\ \mathrm{ff})$
- `iter` $f\ \emptyset = ()$
- `iter` $f\ \{a_1, \ldots, a_n\} = f\ a_1;\ f\ a_2;\ \ldots;\ f\ a_n$
- `singleton` $e \stackrel{\text{def}}{=} \{e\}$
- `subset` $s_1\ s_2 \stackrel{\text{def}}{=} (s_1 \subseteq s_2\ ?\ \mathrm{tt}\ ¿\ \mathrm{ff})$
- `union` $s_1\ s_2 \stackrel{\text{def}}{=} s_1 \cup s_2$

Implementation: sets of values

```
(* cvalues.mli *)
open Values
(* set of machine integers *)
type elt = machine_int
and t
val add : elt -> t -> t
val singleton : elt -> t
val fold : (elt -> 'a -> 'a) -> t -> 'a -> 'a
val iter : (elt -> unit) -> t -> unit
val bot : unit -> t
val isbotempty : unit -> bool
val initerr : unit -> t
val top : unit -> 'a
val join : t -> t -> t
val meet : t -> t -> t
```



```
val leq : t -> t -> bool
val eq : t -> t -> bool
val in_errors : t -> bool
val print : t -> unit
(* forward collecting semantics of arithmetic expressions *)
val f_NAT : string -> t
val f_RANDOM : unit -> t
val f_UMINUS : t -> t
val f_UPLUS : t -> t
val f_PLUS : t -> t -> t
val f_MINUS : t -> t -> t
val f_TIMES : t -> t -> t
val f_DIV : t -> t -> t
val f_MOD : t -> t -> t
```



```
(* cvalues.ml *)
open Values
(* ordered set of machine integers *)
include Set.Make
 (struct
   type t = machine_int
   (* order on values *)
   let compare v1 v2 = match v1, v2 with
   | (ERROR_NAT INITIALIZATION), (ERROR_NAT INITIALIZATION) -> 0
   | (ERROR_NAT INITIALIZATION), (ERROR_NAT ARITHMETIC) -> -1
   | (ERROR_NAT ARITHMETIC), (ERROR_NAT INITIALIZATION) -> 1
   | (ERROR_NAT ARITHMETIC), (ERROR_NAT ARITHMETIC) -> 0
   | (ERROR_NAT e), (NAT i) -> -1
   | (NAT i), (ERROR_NAT e) -> 1
   | (NAT i), (NAT j) -> if (i<j) then -1 else if (i=j) then 0 else 1
  end)
(* infimum *)
```



```
let bot () = empty
(* bottom is emptyset? *)
let isbotempty () = true
(* uninitialization *)
let initerr () = singleton (ERROR_NAT INITIALIZATION)
(* supremum *)
exception ErrorCvalues of string
let top () = raise (ErrorCvalues "top not implemented")
(* least upper bound *)
let join = union
(* greatest lower bound *)
let meet = inter
(* approximation ordering *)
let leq = subset
(* equality *)
let eq = equal
(* included in errors? *)
let in_errors v =
```

```
  let iserror i = match i with
    | (ERROR_NAT e) -> true
    | (NAT j) -> false
  in for_all iserror v
(* printing *)
let print v =
   let printelement e =
      print_machine_int e;
      print_string " "
   in
      print_string "{ ";
     iter printelement v;
      print_string " }"
(* image u s = { u(x) | x in s } *)
let image u s =
    let f e s' = add (u e) s' in
      fold  f s empty
(* set_bin b s1 s2 = { b(x,y) | x in s1 /\ y in s2 } *)
```



```
let set_bin b s1 s2 =
    let f a2 s =
      let g a1 s = add (b a1 a2) s
      in  fold g s1 empty
    in
       fold f s2 empty
(* forward collecting semantics of arithmetic expressions *)
let f_NAT s =
   singleton (machine_int_of_string s)
let r1 = (machine_unary_random ())
let r2 = (machine_unary_random ())
let r3 = (machine_unary_random ())
let f_RANDOM () =
  (* should be the set of all possible values! *)
  add r1 (add r2 (singleton r3))
let f_UMINUS a =
   image machine_unary_minus a
let f_UPLUS a =
```



```
   image machine_unary_plus a
let f_PLUS a1 a2 = set_bin machine_binary_plus a1 a2
let f_MINUS a1 a2 = set_bin machine_binary_minus a1 a2
let f_TIMES a1 a2 = set_bin machine_binary_times a1 a2
let f_DIV a1 a2 = set_bin machine_binary_div a1 a2
let f_MOD a1 a2 = set_bin machine_binary_mod a1 a2
```


Implementation: sets of environments

```
(* cenv.mli *)
open Abstract_Syntax
open Cvalues
open Env
(* set of environments *)
type elt = Env.env
and t
(* infimum *)
val bot : unit -> t
(* check for infimum *)
val is_bot : t -> bool
(* uninitialization *)
val initerr : unit -> t
(* supremum *)
val top : unit -> 'a
```

```
(* copy *)
val copy : t -> t
(* least upper bound *)
val join : t -> t -> t
(* greatest lower bound *)
val meet : t -> t -> t
(* approximation ordering *)
val leq : t -> t -> bool
(* equality *)
val eq : t -> t -> bool
(* printing *)
val print : t -> unit
(* r(X) = {e(X) | X in r} *)
val get : t -> variable -> Cvalues.t
(* r[X <- i] = {e[X <- i] | e in r } *)
val set_elem : t -> variable -> Values.machine_int -> t
(* r[X <- v] = {e[X <- i] | e in r /\ i in v}    *)
val set : t -> variable -> Cvalues.t -> t
```



```
(* collecting semantics of assignment                              *)
(* f_ASSIGN x f r =  {e[x <- i] | e in r /\ i in f({e}) cap I }   *)
val f_ASSIGN : variable -> (t -> Cvalues.t) -> t -> t
(* collecting semantics of boolean expressions                     *)
(* f_EQ f g r =                                                    *)
(*   {e in r | exists v1 in f({e}) cap I: exists v2 in g({e}) cap *)
(*             I: v1 = v2 }                                        *)
val f_EQ : (t -> Cvalues.t) -> (t -> Cvalues.t) -> t -> t
(* f_LT f g r =                                                    *)
(*   {e in R | exists v1 in f({e}) cap I: exists v2 in g({e}) cap *)
(*             I: v1 < v2 }                                        *)
val f_LT : (t -> Cvalues.t) -> (t -> Cvalues.t) -> t -> t
```



```
(* cenv.ml *)
open Variables
open Values
open Cvalues
open Env
(* order on values *)
let compare_values v1 v2 = match v1, v2 with
  | (ERROR_NAT INITIALIZATION), (ERROR_NAT INITIALIZATION) -> 0
  | (ERROR_NAT INITIALIZATION), (ERROR_NAT ARITHMETIC) -> -1
  | (ERROR_NAT ARITHMETIC), (ERROR_NAT INITIALIZATION) -> 1
  | (ERROR_NAT ARITHMETIC), (ERROR_NAT ARITHMETIC) -> 0
  | (ERROR_NAT e), (NAT i) -> -1
  | (NAT i), (ERROR_NAT e) -> 1
  | (NAT i), (NAT j) -> if (i<j) then -1 else if (i=j) then 0 else 1
(* ordered set of environments *)
exception Found of int
include Set.Make
```



```
   (struct
      type t = env
      (* order on environments *)
      let compare r1 r2 =
        try
          for i = 0 to ((number_of_variables ()) - 1) do
            let c = compare_values (get r1 i) (get r2 i) in
              if c != 0 then   raise (Found c)
          done;
          0
        with Found c -> c
   end)
(* infimum *)
let bot () = empty
(* check for infimum *)
let is_bot = is_empty
(* uninitialization *)
let initerr () = singleton (Env.initerr ())
```

```
(* supremum *)
exception ErrorCenv of string
let top () = raise (ErrorCenv "top not implemented")
(* copy *)
let copy s = s (* implementation without side-effects *)
(* least upper bound *)
let join = union
(* greatest lower bound *)
let meet = inter
(* approximation ordering *)
let leq = subset
(* equality *)
let eq = equal
(* printing *)
let print r =
   print_string "{ ";
   let pe e = (print_string "[ ";print_env e;print_string "] ") in
   iter  pe r;
```



```
   print_string "}"
(* r(X) = {e(X) | e in r}               *)
(* val get : t -> variable -> Cvalues.t *)
let get r x =
  let f e s = Cvalues.add (Env.get e x) s in
    fold f r (Cvalues.bot ())
(* r[X <- i] = {e[X <- i] | e in r }                        *)
(* val set_elem : t -> variable -> Values.machine_int -> t *)
let set_elem r x i  =
  let f e s = add (let e' = (Env.copy e) in Env.set e' x i; e') s in
    fold f r empty
(* r[X <- v] = {e[X <- i] | e in r /\ i in v}    *)
(* val set : t -> variable -> Cvalues.t -> t *)
let set r x v =
    let f i s = union (set_elem r x i) s in
      Cvalues.fold f v empty
(* f_ASSIGN x f r =  {e[x <- i] | e in r /\ i in f({e}) cap I } *)
let f_ASSIGN x f r =
```



```
 let assign e s =
   let a i s' =
     match i with
     | ERROR_NAT _ -> s'
     | NAT _ -> add (let e' = (Env.copy e) in (Env.set e' x i; e')) s'
   in  Cvalues.fold a (f (singleton e)) s
 in fold assign r empty
(* cmp c f g r =                                              *)
(*   {e in r | exists v1 in f({e}) cap I: exists v2 in g({e}) *)
(*             cap I: v1 c v2 }                               *)
(* val cmp : (elt -> elt -> Values.machine_bool) -> (t -> t)  *)
(*                                    -> (t -> t) -> t -> t *)
exception Found
let cmp c f g r =
  let isFound i j =
    match (c i j) with
    | ERROR_BOOL _  -> ()
    | BOOLEAN false -> ()
```



```
    | BOOLEAN true  -> raise Found
  in let ok e =
    let s1 = (f (singleton e))  and s2 = (g (singleton e))
      in (try
        let tests2 j =
          (let tests1 i = isFound i j in Cvalues.iter tests1 s1)
        in Cvalues.iter tests2 s2;
           false
        with Found -> true)
    in filter ok r
(* f_EQ f g r =                                               *)
(*   {e in R | exists v1 in f({e}) cap I: exists v2 in g({e}) *)
(*                                       cap I: v1 = v2 } *)
let f_EQ f g r = cmp machine_eq f g r
(* f_LT f g r =                                               *)
(*   {e in R | exists v1 in f({e}) cap I: exists v2 in g({e}) *)
(*                                       cap I: v1 < v2 } *)
let f_LT f g r = cmp machine_lt f g r
```

Implementation: Forward collecting semantics of arithmetic expressions

```
(* caexp.mli *)
open Abstract_Syntax
open Cvalues
open Cenv
(* evaluation of arithmetic operations *)
val c_aexp : aexp -> Cenv.t -> Cvalues.t
```



```
(* caexp.ml *)
open Abstract_Syntax
(* evaluation of arithmetic operations *)
let rec c_aexp a r = match a with
  | (Abstract_Syntax.NAT i) -> (Cvalues.f_NAT i)
  | (VAR v)       -> (Cenv.get r v)
  | RANDOM        -> Cvalues.f_RANDOM ()
  | (UPLUS a1)    -> (Cvalues.f_UPLUS (c_aexp a1 r))
  | (UMINUS a1)   -> (Cvalues.f_UMINUS (c_aexp a1 r))
  | (PLUS (a1, a2))  -> (Cvalues.f_PLUS (c_aexp a1 r) (c_aexp a2 r))
  | (MINUS (a1, a2)) -> (Cvalues.f_MINUS (c_aexp a1 r) (c_aexp a2 r))
  | (TIMES (a1, a2)) -> (Cvalues.f_TIMES (c_aexp a1 r) (c_aexp a2 r))
  | (DIV (a1, a2))   -> (Cvalues.f_DIV (c_aexp a1 r) (c_aexp a2 r))
  | (MOD (a1, a2))   -> (Cvalues.f_MOD (c_aexp a1 r) (c_aexp a2 r))
```


Implementation: Forward collecting semantics of boolean expressions

```
(* cbexp.mli *)
open Abstract_Syntax
open Cvalues
open Cenv
(* evaluation of boolean operations *)
val c_bexp : bexp -> Cenv.t -> Cenv.t
```



```
(* cbexp.ml *)
open Abstract_Syntax
open Cvalues
open Cenv
open Caexp
(* evaluation of boolean operations *)
let rec c_bexp b r =
  match b with
  | TRUE           -> r
  | FALSE          -> (Cenv.bot ())
  | (EQ (a1, a2))  -> f_EQ (c_aexp a1) (c_aexp a2) r
  | (LT (a1, a2))  -> f_LT (c_aexp a1) (c_aexp a2) r
  | (AND (b1, b2)) -> Cenv.meet (c_bexp b1 r) (c_bexp b2 r)
  | (OR (b1, b2))  -> Cenv.join (c_bexp b1 r) (c_bexp b2 r)
```

We have made an oversimplification in the last alternative ignoring the case when `b1` holds and `b2` yields an error.

Implementation: fixpoints

```
(* fixpoint.mli *)
open Cenv
(* lfp x c f : iterative computation of the c-least fixpoint of f *)
(* c-greater than or equal to the prefixpoint x (f(x) >= x)       *)
val lfp : t -> (t -> t -> bool) -> (t -> t) -> t
```



```
(* fixpoint.ml *)
open Cenv
(* lfp x c f : iterative computation of the c-least fixpoint of f *)
(* c-greater than or equal to the prefixpoint x (f(x) >= x)       *)
(* x0 = x; ...; xn+1 = xn U f(xn);... ; xl where xl c xl U f(xl)  *)
let rec lfp x c f =
  let x' = (join x (f (copy x))) in
    if (c x' x) then x'
      else lfp x' c f
```


Note on the iterates: if the random choice picks 3 (may be) different values at each choice then f may not be monotonic. It follows that the iteration sequence

- $x^0 = x$
- $x^{n+1} = f(x^n)$ if $f(x^n) \not\sqsubseteq x^n$
- $x^{n+1} = x_n$ if $f(x^n) = x^n$

may not converge, even if $x \sqsubseteq f(x)$. So we compute instead

- $x^{n+1} = \bigcup_{k \leq n} f(x^k)$ if $f(x^n) \not\sqsubseteq x^n$

which is extensive and ultimately convergent to $\bigcup_{n\in\mathbb{N}} f(x^n)$. The two iterations are the same when initially x is a prefixpoint and f is monotonic. Convergence is assumed to follow from other considerations (otherwise the computation may not terminate properly).

To trace the fixpoint iterates:

```
(* fixpoint.ml *)
open Cenv
(* lfp x c f : iterative computation of the c-least fixpoint of f *)
(* c-greater than or equal to the prefixpoint x (f(x) >= x)       *)
(* x0 = x; ...; xn+1 = xn U f(xn);... ; xl where xl c xl U f(xl)  *)
let lfp x c f =
  let rec iterate n x =
     print_string "iterate ";print_int n;print_string " = ";print x;
     print_newline ();
     let x' = (join x (f (copy x))) in
        (if (c x' x) then (print_string "fixpoint = ";print x';
                            print_newline ();x')
          else iterate (n + 1) x')
   in iterate 0 x
```

Implementation: forward reachability collecting semantics of commands

```
(* ccom.mli *)
open Abstract_Syntax
open Labels
open Cenv
(* forward collecting semantics of commands *)
val ccom : com -> Cenv.t -> label -> Cenv.t

(* ccom.ml *)
open Abstract_Syntax
open Labels
open Cenv
open Caexp
open Cbexp
open Fixpoint
```



```
(* collecting semantics of commands *)

exception Error of string
let rec ccom c r l =
  match c with
  | (SKIP (l', l'')) ->
      if (l = l') then r
      else if (l = l'') then r
      else (raise (Error "SKIP incoherence"))
  | (ASSIGN (l',x,a,l'')) ->
      if (l = l') then r
      else if (l = l'') then
           f_ASSIGN x (c_aexp a) r
      else (raise (Error "ASSIGN incoherence"))
  | (SEQ (l', s, l'')) ->
      (ccomseq s r l)
  | (IF (l', b, nb, t, f, l'')) ->
      (if (l = l') then r
```



```
       else if (incom l t) then
          (ccom t (c_bexp b r) l)
       else if (incom l f) then
          (ccom f (c_bexp nb r) l)
       else if (l = l'') then
          (join (ccom t (c_bexp b r) (after t))
                (ccom f (c_bexp nb r) (after f)))
       else (raise (Error "IF incoherence")))
  | (WHILE (l', b, nb, c', l'')) ->
     let f x = join r (ccom c' (c_bexp b x) (after c'))
     in let i = lfp (bot ()) leq f in
       (if (l = l') then i
        else if (incom l c') then (ccom c' (c_bexp b i) l)
        else if (l = l'') then (c_bexp nb i)
        else (raise (Error "WHILE incoherence")))
and ccomseq s r l = match s with
  | []   -> raise (Error "empty SEQ incoherence")
  | [c]  -> if (incom l c) then (ccom c r l)
```



```
            else (raise (Error "SEQ incoherence"))
  | h::t -> if (incom l h) then (ccom h r l)
            else (ccomseq t (ccom h r (after h)) l)
```

Implementation: forward reachability collecting interpretor

```
(* main.ml *)
open Program_To_Abstract_Syntax
open Labels
open Pretty_Print
open Cenv
open Ccom
let _ =
  let arg = if (Array.length  Sys.argv) = 1 then ""
            else Sys.argv.(1) in
     Random.self_init ();
    let p = (abstract_syntax_of_program arg) in
      (print (initerr ());
       pretty_print p;
       print (ccom p (initerr ()) (after p));
       print_newline ())
```


Implementation: makefile

```
# makefile

SOURCES = \
symbol_Table.mli \
symbol_Table.ml \
variables.mli \
variables.ml \
abstract_Syntax.ml \
concrete_To_Abstract_Syntax.mli \
concrete_To_Abstract_Syntax.ml \
labels.mli \
labels.ml \
parser.mli \
parser.ml \
lexer.ml \
```



```
program_To_Abstract_Syntax.mli \
program_To_Abstract_Syntax.ml \
pretty_Print.mli \
pretty_Print.ml \
values.mli \
values.ml \
cvalues.mli \
cvalues.ml \
env.mli \
env.ml \
cenv.mli \
cenv.ml \
caexp.mli \
caexp.ml \
cbexp.mli \
cbexp.ml \
fixpoint.mli \
fixpoint.ml \
```



```
ccom.mli \
ccom.ml \
main.ml

.PHONY : help
help :
   @echo ""
   @echo "make help          : this help"
   @echo "make trace         : trace fixpoint iterates"
   @echo "make untrace       : don't trace fixpoint iterates"
   @echo "make compile       : compile"
   @echo "./a.out filename : execute"
   @echo "make examples      : execute the examples"
   @echo "make errors        : execute the examples with runtime errors"
   @echo "make clean         : remove auxilairy files"
   @echo ""

.PHONY : trace preparetrace
```

```
trace: preparetrace compile
   @echo "fixpoint tracing mode"
preparetrace:
   @/bin/rm -f fixpoint.ml
   @ln -s fixpoint_printing_iterates.ml fixpoint.ml

.PHONY : untrace prepareuntrace
untrace: prepareuntrace compile
   @echo "no fixpoint tracing, recompile!"
prepareuntrace:
   @/bin/rm -f fixpoint.ml
   @ln -s fixpoint_no_printing.ml fixpoint.ml

.PHONY : compile
compile:
   ocamlyacc parser.mly
   ocamllex lexer.mll
#   ocamlc -i $(SOURCES) # to print types
```



```
   ocamlc $(SOURCES)

.PHONY : examples
examples :
   ./a.out ../Examples/example00.sil
   ./a.out ../Examples/example01.sil
   ./a.out ../Examples/example02.sil
   ./a.out ../Examples/example03.sil
   ./a.out ../Examples/example04.sil
   ./a.out ../Examples/example05.sil
   ./a.out ../Examples/example07.sil

.PHONY : errors
errors :
   ./a.out ../Examples/example06.sil
   ./a.out ../Examples/example08.sil
   ./a.out ../Examples/example09.sil
   ./a.out ../Examples/example10.sil
```



```
   ./a.out ../Examples/example11.sil

.PHONY :
clean :
   /bin/rm -f *.cmi *.cmo *~ a.out lexer.ml parser.mli parser.ml
```


Implementation: Examples

```
Script started on Mon Apr  4 15:22:39 2005
% make clean
...
% make untrace
...
% make compile
...
% make examples

./a.out ../Examples/example0.sil
{ [ ] }
:
  skip
:

```

```
{ [ ] }
./a.out ../Examples/example1.sil
{ [ x = _O_(i); ] }
:
  x := 1;
:
  while (x < 100) do
    2:
      x := (x + 1)
    3:
  od {((100 < x) | (x = 100))}
:

{ [ x = 100; ] }
./a.out ../Examples/example2.sil
{ [ x = _O_(i); y = _O_(i); ] }
:
  x := (-1073741823 - 1);
```



```
:
  y := (x - 1)
:

{ }
./a.out ../Examples/example3.sil
{ [ x = _O_(i); y = _O_(i); ] }
:
  x := 0;
:
  y := 1
:

{ [ x = 0; y = 1; ] }
./a.out ../Examples/example4.sil
{ [ x = _O_(i); ] }
```



```
:
  if true then
    1:
      x := 1
    2:
  else {false}
    3:
      x := 0
    4:
  fi
:

{ [ x = 1; ] }
./a.out ../Examples/example5.sil
{ [ x = _O_(i); ] }
:
  if false then
    1:
```



```
      x := 1
    2:
  else {true}
    3:
      x := 0
    4:
  fi
:

{ [ x = 0; ] }
./a.out ../Examples/example7.sil
{ [ x = _O_(i); ] }
:
  x := 1;
:
  while ((x < 10) | (x = 10)) do
    2:
      x := (x + 1)
```

```
    3:
  od {(10 < x)}
:

{ [ x = 11; ] }
% ^Dexit

Script done on Mon Apr  4 15:23:08 2005
```


Implementation: Examples of runtime errors than stop execution

```
Script started on Mon Apr  4 15:23:25 2005
% make errors

./a.out ../Examples/example6.sil
{ [ x = _O_(i); ] }
:
  x := -1073741824
:

{ }
./a.out ../Examples/example8.sil
{ [ x = _O_(i); ] }
:
```



```
  x := 1073741823
:

{ [ x = 1073741823; ] }
./a.out ../Examples/example9.sil
{ [ x = _O_(i); y = _O_(i); z = _O_(i); t = _O_(i); ] }
:
  x := (-536870912 * 2);
:
  y := (536870912 * 2);
:
  z := ((-1073741823 - 1) * 1);
:
  t := ((-1073741823 - 1) * 1073741823)
:

{ }
./a.out ../Examples/example10.sil
```



```
{ [ x = _O_(i); ] }
:
  x := ?;
:
  if (x < (-1073741823 - 1)) then
    2:
      x := 1
    3:
  else {(((-1073741823 - 1) < x) | (x = (-1073741823 - 1)))}
    4:
      x := 0
    5:
  fi
:

{ [ x = 0; ] }
./a.out ../Examples/example11.sil
{ [ x = _O_(i); ] }
```

```
:
  x := 1;
:
  while (0 < 1073741824) do
    2:
      x := (x + 1)
    3:
  od {((1073741824 < 0) | (1073741824 = 0))}
:

{ }
% ^Dexit

Script done on Mon Apr  4 15:23:36 2005
```


Conclusion

– We have examplified the calculational design of program static analyzers by abstract interpretation of a formal semantics;
– This provides a thorough understanding of the abstraction process allowing for the later development of useful large scale analyzers;
– Scales up manually by small parts;
– One can hope that in the future one can extract analyzers from correctness proofs;

– Since such extracted will probably be inefficient, one can consider:
  - The use of manually constructed static analyzers to compute inductive fixpoint approximations (which involve iterations with convergence acceleration)
  - The use of static analyzers extracted from the correctness proof to check that the previous fixpoint approximations are indeed inductive (which involves no iteration)

Bibliography

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The Calculational Design of a Generic Abstract Interpreter. In *Calculational System Design*, M. Broy & R. Steinbrüggen, editors, NATO ASI Series F. IOS Press, Amsterdam, 1999. http://www.dmi.ens.fr/~cousot/COUSOTpapers/Marktoberdorf98.shtml, November 1998.

[2] Patrick Cousot.
The Marktoberdorf'98 generic abstract interpreter. http://www.dmi.ens.fr/~cousot/Marktoberdorf98.shtml, November 1998.

THE END

My MIT web site is http://www.mit.edu/~cousot/

The course web site is http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/.