« Mathematical foundations: (7) Approximation »

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THESE

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par

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METHODES ITERATIVES DE CONSTRUCTION ET D'APPROXIMATION DE POINTS FIXES D'OPERATEURS MONOTONES SUR UN TREILLIS, ANALYSE SEMANTIQUE DES PROGRAMMES.

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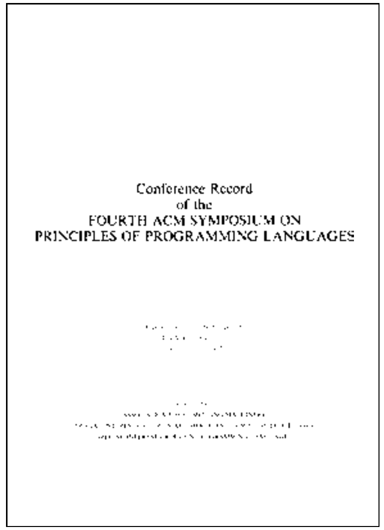

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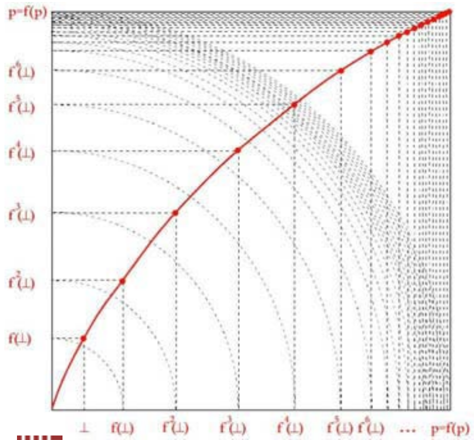
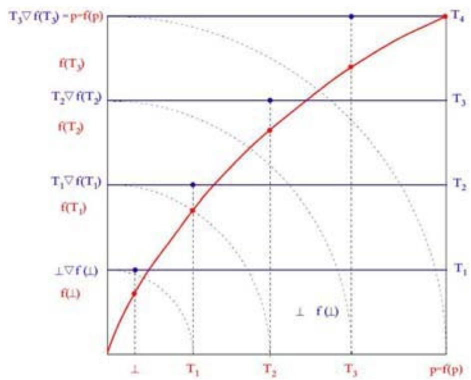
Intuition for the iterative fixpoint computation of monotone/extensive operators (in general non convergent)



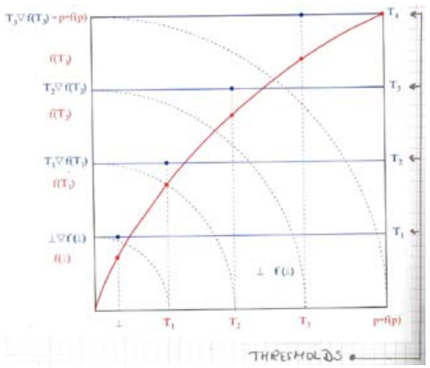
In general the iterates $\bot$, $f(\bot)$, ..., $f^n(\bot)$, ... are not convergent or converge mathematically in infinitely many steps.

Intuition for the iterative fixpoint computation with convergence acceleration by widening of monotone/extensive operators (with overapproximation)



The convergence of $\bot$, $f(\bot), \ldots, f^n(\bot), \ldots$ is accelerated as $x^0 \stackrel{\text{def}}{=} \bot, \ldots, x^{n+1} \stackrel{\text{def}}{=} x^x \,\nabla\, f^n(\bot), \ldots$ using a widening ∇.

Example of widening using finitely many thresholds



In this example, the widening is defined using thresholds $\{T_1, T_2, \ldots, T_n\}$ in finite number such that

- The thresholds include the supremum;
- $x \nabla y$ is the least threshold T_i greater than or equal to both x and y.

Approximate fixpoint abstraction

À la Kleene, Galois connection based, continuous transformer, fixpoint approximation

If
- $L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ is a complete lattice;
- $F \in L \stackrel{\text{uc}}{\longmapsto} L$ is continuous for $\sqsubseteq$;
- $L^\sharp(\sqsubseteq^\sharp, \bot^\sharp, \top^\sharp, \sqcup^\sharp, \sqcap^\sharp)$ is a complete lattice;
- $L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} L^\sharp$ is a Galois connection;
- $F^\sharp \in L^\sharp \mapsto L^\sharp$ is continuous for $\sqsubseteq^\sharp$;
- $\alpha \circ F \circ \gamma \sqsubseteq^\sharp F^\sharp$;

then:

$$\mathsf{lfp}\ F \sqsubseteq \gamma(\mathsf{lfp}\ F^\sharp)$$

PROOF. 1. α is monotonic:

$f \sqsubseteq g$	[hypothesis]
$\Longrightarrow f \sqsubseteq g \ \wedge\ \alpha(g) \sqsubseteq \alpha(g)$	[reflexivity]
$\Longrightarrow f \sqsubseteq g \ \wedge\ g \sqsubseteq \gamma(\alpha(g))$	[Galois connection]
$\Longrightarrow f \sqsubseteq \gamma(\alpha(g))$	[transitivity]
$\Longrightarrow \alpha(f) \sqsubseteq^\sharp \alpha(g)$	[Galois connection]

2.

$\bot \sqsubseteq \gamma(\bot^\sharp)$	[infimum]
$\Longrightarrow F^0(\bot) \sqsubseteq \gamma({F^\sharp}^0(\bot^\sharp))$	[def. iterates]
$\Longrightarrow \alpha(F^0(\bot)) \sqsubseteq^\sharp {F^\sharp}^0(\bot^\sharp)$	[Galois connection]

3. $\alpha(F^n(\bot)) \sqsubseteq^\sharp F^{\sharp^n}(\bot^\sharp)$ [induction hypothesis]

$\Longrightarrow F^n(\bot) \sqsubseteq \gamma(F^{\sharp^n}(\bot^\sharp))$ [Galois connection]

$\Longrightarrow F(F^n(\bot)) \sqsubseteq F \circ \gamma(F^{\sharp^n}(\bot^\sharp))$ [F monotonic]

$\Longrightarrow \alpha(F(F^n(\bot))) \sqsubseteq^\sharp \alpha \circ F \circ \gamma(F^{\sharp^n}(\bot^\sharp))$ [α monotonic 1.]

$\alpha \circ F \circ \gamma(F^{\sharp^n}(\bot^\sharp)) \sqsubseteq^\sharp F^\sharp(F^{\sharp^n}(\bot^\sharp))$ [hypothesis]

$\Longrightarrow \alpha(F(F^n(\bot))) \sqsubseteq^\sharp F^\sharp(F^{\sharp^n}(\bot^\sharp))$ [transitivity]

$\Longrightarrow \alpha(F^{n+1}(\bot)) \sqsubseteq^\sharp F^{\sharp^{n+1}}(\bot^\sharp)$ [def. iterates]

4. $\forall n : \alpha(F^n(\bot)) \sqsubseteq^\sharp F^{\sharp^n}(\bot^\sharp)$ [2., 3., recurrence]

$\Longrightarrow \forall n : \alpha(F^n(\bot)) \sqsubseteq^\sharp \sqcup^\sharp_{m\geq 0} F^{\sharp^m}(\bot^\sharp)$ [lub]

$\Longrightarrow \alpha(F^0(\bot)) \sqsubseteq^\sharp F^{\sharp^0}(\bot^\sharp)$ [Galois connection]

$\Longrightarrow \forall n : \alpha(F^n(\bot)) \sqsubseteq^\sharp \mathsf{lfp}\, F^\sharp$ [Tarski constructive th.]

$\Longrightarrow \forall n : F^n(\bot) \sqsubseteq^\sharp \gamma(\mathsf{lfp}\, F^\sharp)$ [Galois connection]

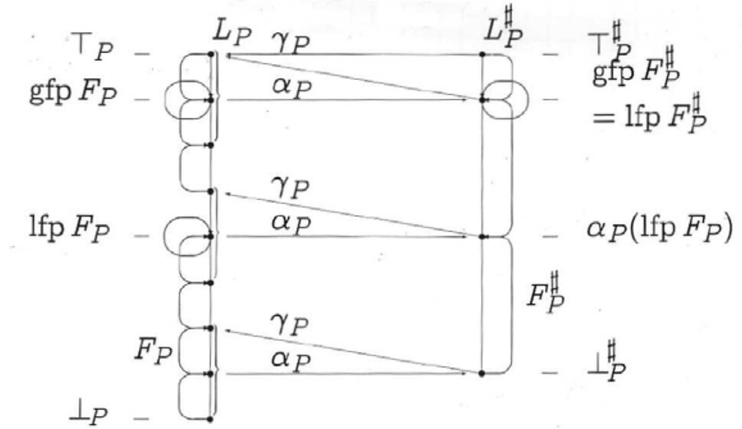
$\Longrightarrow \sqcup_{n\geq 0} F^n(\bot) \sqsubseteq^\sharp \gamma(\mathsf{lfp}\, F^\sharp)$ [least upper bound]

$\Longrightarrow \mathsf{lfp}\,(F) \sqsubseteq^\sharp \gamma(\mathsf{lfp}\, F^\sharp)$ [Tarski constructive th.]

□

Note: we need $\alpha \circ F \circ \gamma(X) \sqsubseteq^\sharp F^\sharp(X)$ only when $X = F^{\sharp^n}(\bot^\sharp)$, $n \in \mathbb{N}$ and so we can relax the hypothesis and assume e.g. $\forall X \sqsubseteq^\sharp \mathsf{lfp}\, F^\sharp : \alpha \circ F \circ \gamma(X) \sqsubseteq^\sharp F^\sharp(X)$

Example:



À la Kleene, Galois connection, monotone transformer-based fixpoint approximation

If
- $\langle L, \sqsubseteq, \sqcup\rangle$ is a cpo
- $F \in L \stackrel{m}{\longmapsto} L$ is monotonic for $\sqsubseteq$
- $a \in L$ is a prefixpoint of F, i.e.: $a \sqsubseteq F(a)$
- $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup}\rangle$ is a cpo
- $\overline{F} \in \overline{L} \stackrel{m}{\longmapsto} \overline{L}$ is monotonic for $\overline{\sqsubseteq}$
- $\langle L, \sqsubseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{L}, \overline{\sqsubseteq}\rangle$ is a Galois connection
- $[\forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)}\, \overline{F} \Longrightarrow \alpha \circ F \circ \gamma(y) \mathrel{\overline{\sqsubseteq}} \overline{F}(y)]$

$\Longleftrightarrow [\forall x \in L : \alpha(x) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)}\, \overline{F} \Longrightarrow F(x) \sqsubseteq \gamma \circ \overline{F} \circ \alpha(x)]$

$$\Longleftrightarrow \quad [\forall x \in L : \alpha(x) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F(x) \mathrel{\overline{\sqsubseteq}} \overline{F} \circ \alpha(x)]$$

$$\Longleftrightarrow \quad [\forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F \circ \gamma(y) \sqsubseteq \gamma \circ \overline{F}(y)]$$

then $\quad \mathsf{lfp}^{\sqsubseteq}_{a} F \sqsubseteq \gamma(\mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F})$

PROOF. – The equivalence of the different statements of overapproximation of F by $\overline{F}$ can be proved as follows:

$\forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F \circ \gamma(y) \mathrel{\overline{\sqsubseteq}} \overline{F}(y)$

$\Longrightarrow \forall x \in L : \alpha(x) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F \circ \gamma(\alpha(x)) \mathrel{\overline{\sqsubseteq}} \overline{F}(\alpha(x))$ $\wr$by letting $y = \alpha(x)$$\int$

$\Longrightarrow \forall x \in L : \alpha(x) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F(x) \mathrel{\overline{\sqsubseteq}} \overline{F} \circ \alpha(x)$ $\wr\gamma \circ \alpha$ is extensive, F and α are monotone, def. composition $\circ\int$

$\Longrightarrow \forall x \in L : \alpha(x) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F(x) \sqsubseteq \gamma \circ \overline{F} \circ \alpha(x)$ $\wr$def. Galois connection$\int$

$\Longrightarrow \forall y \in \overline{L} : \alpha(\gamma(y)) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F(\gamma(y)) \sqsubseteq \gamma \circ \overline{F} \circ \alpha(\gamma(y))$ $\wr$by letting $x = \gamma(y)$$\int$

$\Longrightarrow \forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F(\gamma(y)) \sqsubseteq \gamma \circ \overline{F} \circ \alpha(\gamma(y))$ $\wr$since $\alpha \circ \gamma$ is reductive in a Galois connection and so $y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F}$ implies $\alpha(\gamma(y)) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F}$ by transitivity$\int$

$\Longrightarrow \forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F \circ \gamma(y) \sqsubseteq \gamma \circ \overline{F} \circ \alpha \circ \gamma(y)$ $\wr$def. composition $\circ\int$

$\Longrightarrow \forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow F \circ \gamma(y) \sqsubseteq \gamma \circ \overline{F}(y)$ $\wr\alpha \circ \gamma$ is reductive, γ, $\overline{F}$ monotone, transitivity$\int$

$\Longrightarrow \forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F \circ \gamma(y) \mathrel{\overline{\sqsubseteq}} \alpha \circ \gamma \circ \overline{F}(y)$ $\wr\alpha$ is monotone$\int$

$\Longrightarrow \forall y \in \overline{L} : y \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} \Longrightarrow \alpha \circ F \circ \gamma(y) \mathrel{\overline{\sqsubseteq}} \overline{F}(y)$ $\wr\alpha \circ \gamma$ is reductive, transitivity$\int$

- We let $\langle F^{\delta}, \delta \in \mathbb{O}\rangle$ be the iterates of F starting from a. By the hypothesis that $\langle L, \sqsubseteq, \sqcup\rangle$ is a cpo, F is monotonic and $a \sqsubseteq F(a)$, they are a well-defined increasing chain and an ordinal ϵ such that $\mathsf{lfp}^{\sqsubseteq}_{a} F = F^{\epsilon}$.
- We let $\langle \overline{F}^{\delta}, \delta \in \mathbb{O}\rangle$ be the iterates of $\overline{F}$ starting from $\alpha(a)$. Observe that $\alpha(a) \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F}$ and so $\alpha \circ F(a) \mathrel{\overline{\sqsubseteq}} \overline{F} \circ \alpha(a)$. We have $\alpha \circ F \mathrel{\dot{\overline{\sqsubseteq}}} \overline{F} \circ \alpha$, $F(a) \sqsupseteq a$ and α is monotone by $\langle L, \sqsubseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{L}, \overline{\sqsubseteq}\rangle$ and so $\overline{F}(\alpha(a)) \mathrel{\overline{\sqsupseteq}} \alpha(F(a)) \mathrel{\overline{\sqsupseteq}} \alpha(a)$ proving $\alpha(a) \sqsubseteq \overline{F}(a)$.
  $\alpha(a)$ is a prefixpoint of the monotone operator $\overline{F}$ on the cpo $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup}\rangle$ proving, as shown in the constructive version of Tarski's fixpoint theorem, that they are a well-defined increasing chain and an ordinal ϵ' such that $\mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F} = \overline{F}^{\epsilon'}$ [1].
- We have
  - $F^{0} = a \sqsubseteq \gamma \circ \alpha(a) = \gamma(\overline{F}^{0})$

[1] Indeed $\langle L, \sqsubseteq, \sqcup\rangle$ and $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup}\rangle$ need only be $\max(\epsilon, \epsilon')$-cpos.

- If $F^{\delta} \sqsubseteq \gamma(\overline{F}^{\delta})$ by induction hypothesis, then $\overline{F}^{\delta} \mathrel{\overline{\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F}$ and so $F^{\delta+1} = F(F^{\delta}) \sqsubseteq F \circ \gamma(\overline{F}^{\delta}) \sqsubseteq \gamma \circ \overline{F}(\overline{F}^{\delta}) = \gamma(\overline{F}^{\delta+1})$
- If λ is a limit ordinal and, by induction hypothesis, $\forall \delta < \lambda : F^{\delta} \sqsubseteq \gamma(\overline{F}^{\delta})$,

then $\alpha(F^{\delta}) \mathrel{\overline{\sqsubseteq}} \overline{F}^{\delta}$ and so

$$\alpha(F^{\lambda}) = \alpha(\bigsqcup_{\beta<\lambda} F^{\beta}) = \overline{\bigsqcup_{\beta<\lambda}} \alpha(F^{\beta}) \quad \wr\text{since } \alpha \text{ is a complete join morphism}\int$$

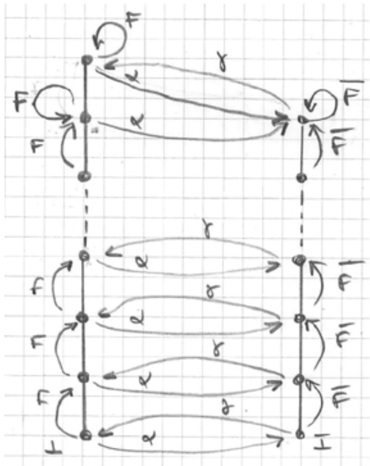
$$\mathrel{\overline{\sqsubseteq}} \overline{\bigsqcup_{\beta<\lambda}} \overline{F}^{\beta} = \overline{F}^{\lambda}$$

and so $F^{\lambda} \sqsubseteq \gamma(F^{\lambda})$.

- By transfinite induction, $\forall \delta \in \mathbb{O} : F^{\delta} \sqsubseteq \gamma(\overline{F}^{\delta})$.
- Finally, $\mathsf{lfp}^{\sqsubseteq}_{a} F = F^{\epsilon} = F^{\max(\epsilon,\epsilon')} \sqsubseteq \gamma(F^{\max(\epsilon,\epsilon')} = \gamma(F^{\epsilon'}) = \gamma(\mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)} \overline{F})$.

□

Example:



These theorems are used (respectively for continuous and monotone functions) in presence of best approximation when α selects the best possible abstraction.

Soundness and (in-)completeness of abstractions

To prove $\mathsf{lfp}_a^{\sqsubseteq} F \sqsubseteq P$, where the fixpoint or invariants are uncomputable, we must overapproximate $\mathsf{lfp}_a^{\sqsubseteq} F$ and underapproximate P. Since in practice this is very hard in non-trivial cases, we choose the abstract domain $\overline{L}$ to be expressive enough to express the properties $P = \gamma(\overline{P})$ to be proved. By the previous theorems, we get:

– Soundness:

$$\mathsf{lfp}_{\alpha(a)}^{\overline{\sqsubseteq}} \overline{F} \mathrel{\overline{\sqsubseteq}} \overline{P} \Longrightarrow \mathsf{lfp}_a^{\sqsubseteq} F \sqsubseteq \gamma(\overline{P})$$

We have also seen previously that the additional commutation condition $\overline{F} \circ \alpha = \alpha \circ F$ implies $\overline{F} = \alpha \circ F \circ \gamma$, $\alpha(\mathsf{lfp}_a^{\sqsubseteq} F) = \mathsf{lfp}_{\alpha(a)}^{\overline{\sqsubseteq}} \overline{F}$ and $\epsilon' \leq \epsilon$, so, in that case, we have

– Completeness:

$$\mathsf{lfp}_a^{\sqsubseteq} F \sqsubseteq \gamma(\overline{P}) \Longrightarrow \mathsf{lfp}_{\alpha(a)}^{\overline{\sqsubseteq}} \overline{F} \mathrel{\overline{\sqsubseteq}} \overline{P}$$

In case of incompleteness, the only way to get more precise abstractions is therefore to refine the abstract transformer $\overline{F}$ (choosing $\overline{F} = \alpha \circ F \circ \gamma$ instead of $\overline{F} \mathrel{\dot{\overline{\sqsupset}}} \alpha \circ F \circ \gamma$) and otherwise to refine the abstraction α.

À la Kleene, continuous abstraction function-based fixpoint approximation

If
- $\langle L, \sqsubseteq, \sqcup \rangle$ is a cpo
- $F \in L \overset{m}{\longmapsto} L$ is monotonic for $\sqsubseteq$
- $a \in L$ is a prefixpoint of F, i.e.: $a \sqsubseteq F(a)$
- $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup} \rangle$ is a cpo
- $\overline{F} \in \overline{L} \overset{m}{\longmapsto} \overline{L}$ is monotonic for $\overline{\sqsubseteq}$
- $\alpha \in L \overset{uc}{\longmapsto} \overline{L}$ is *upper-continuous*
- $\alpha \circ F \mathrel{\dot{\overline{\sqsubseteq}}} \overline{F} \circ \alpha$

then

$$\alpha(\mathsf{lfp}_a^{\sqsubseteq} F) \mathsf{lfp}_{\alpha(a)}^{\overline{\sqsubseteq}} \overline{F}$$

PROOF. – We let $\langle F^\delta,\ \delta \in \mathbb{O}\rangle$ be the iterates of F starting from a. By the hypothesis that $\langle L,\ \sqsubseteq,\ \sqcup\rangle$ is a cpo, F is monotonic and $a \sqsubseteq F(a)$, they are a well-defined increasing chain and there exists an ordinal ϵ such that $\mathsf{lfp}^{\sqsubseteq}_a\ F = F^\epsilon$.

– We let $\langle \overline{F}^\delta,\ \delta \in \mathbb{O}\rangle$ be the iterates of $\overline{F}$ starting from $\alpha(a)$. We have $\overline{F}(\alpha(a)) \mathrel{\overline{\sqsupseteq}} \alpha(F(a)) \mathrel{\overline{\sqsupseteq}} \alpha(a)$ since $\alpha \circ F \mathrel{\dot{\overline{\sqsubseteq}}} \overline{F} \circ \alpha$, $F(a) \sqsupseteq a$ and α is upper-continuous whence monotone. So $\alpha(a)$ is a prefixpoint of the monotonic operator $\overline{F}$ on the cpo $\langle \overline{L},\ \overline{\sqsubseteq},\ \overline{\sqcup}\rangle$ proving, as shown in the constructive version of Tarski's fixpoint theorem, that they are a well-defined increasing chain and there exists an ordinal ϵ' such that $\mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)}\ \overline{F} = \overline{F}^{\epsilon'}$ [2].

– We have

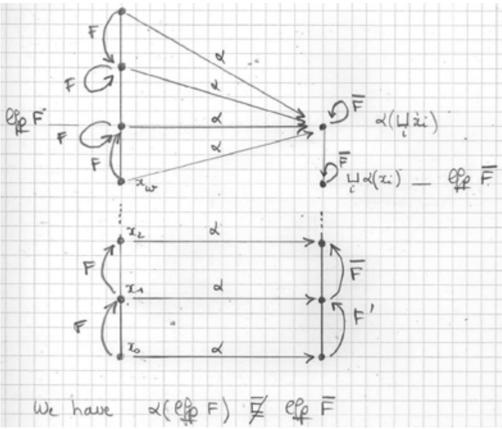
- $\alpha(F^0) = \alpha(a) = \overline{F}^0$
- If $\alpha(F^\delta) \mathrel{\overline{\sqsubseteq}} \overline{F}^\delta$ by induction hypothesis, then
  $\alpha(F^{\delta+1}) = \alpha(F(F^\delta)) \mathrel{\overline{\sqsubseteq}} \overline{F}(\alpha(F^\delta)) \mathrel{\overline{\sqsubseteq}} \overline{F}(\overline{F}^\delta) = \overline{F}^{\delta+1}$

[2] Again, $\langle L,\ \sqsubseteq,\ \sqcup\rangle$ and $\langle \overline{L},\ \overline{\sqsubseteq},\ \overline{\sqcup}\rangle$ need only be $\max(\epsilon, \epsilon')$-cpos.

- If λ is a limit ordinal and, by induction hypothesis, $\forall \delta < \lambda : \alpha(F^\delta) \mathrel{\overline{\sqsubseteq}} \overline{F}^\delta$ then
  $\alpha(F^\lambda) = \alpha(\bigsqcup_{\beta<\lambda} F^\beta) = \overline{\bigsqcup_{\beta<\lambda}} \alpha(F^\beta)$ ≀since $\langle F^\delta,\ \delta < \lambda\rangle$ is an increasing chain and α is a upper-continuous≀
  $\mathrel{\overline{\sqsubseteq}} \overline{\bigsqcup_{\beta<\lambda}} \overline{F}^\beta = \overline{F}^\lambda$
- By transfinite induction, $\forall \delta \in \mathbb{O} : \alpha(F^\delta) \mathrel{\overline{\sqsubseteq}} \overline{F}^\delta$.
- In conclusion, $\alpha(\mathsf{lfp}^{\sqsubseteq}_a\ F) = \alpha((F^\epsilon) = \alpha(F^{\max(\epsilon,\epsilon')}) \mathrel{\overline{\sqsubseteq}} \overline{F}^{\max(\epsilon,\epsilon')} = \overline{F}^{\epsilon'} = \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)}\ \overline{F}$.

□

A counter-example showing the continuity of the abstraction function is necessary



We have $\alpha(\mathsf{lfp}^{\sqsubseteq}_a\ F) \mathrel{\overline{\not\sqsubseteq}} \mathsf{lfp}^{\overline{\sqsubseteq}}_{\alpha(a)}\ \overline{F}$ where $a = \bot$ and $\alpha(a) = \overline{\bot}$.

This theorem is used in absence of best approximation when α selects among possible (minimal) abstractions

À la Kleene, monotone concretization-based fixpoint approximation

If
- $\langle L,\ \sqsubseteq,\ \sqcup\rangle$ is a cpo
- $F \in L \stackrel{m}{\longmapsto} L$ is monotonic for $\sqsubseteq$
- $\langle \overline{L},\ \overline{\sqsubseteq},\ \overline{\sqcup}\rangle$ is a cpo
- $\overline{F} \in \overline{L} \stackrel{m}{\longmapsto} \overline{L}$ is monotonic for $\overline{\sqsubseteq}$
- $\overline{a} \in \overline{L}$ is a prefixpoint of $\overline{F}$, i.e.: $\overline{a} \mathrel{\overline{\sqsubseteq}} \overline{F}(\overline{a})$
- $\gamma \in \overline{L} \stackrel{m}{\longmapsto} L$ is monotonic
- $\gamma(\overline{a})$ is a prefixpoint of $\overline{a}$, i.e. $\gamma(\overline{a}) \sqsubseteq F(\gamma(\overline{a}))$
- $F \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ \overline{F}$

then $\mathsf{lfp}^{\sqsubseteq}_{\gamma(\overline{a})}\ F \sqsubseteq \gamma(\mathsf{lfp}^{\overline{\sqsubseteq}}_{\overline{a}}\ \overline{F})$

Proof. – Observe that the hypotheses that $\overline{a}$ and $\gamma(\overline{a})$ are respective prefixpoints of $\overline{F}$ and F are independent, as shown by the following examples:

– We let $\langle F^\delta, \delta \in \mathbb{O}\rangle$ be the iterates of F starting from $\gamma(\overline{a})$. By the hypothesis that $\langle L, \sqsubseteq, \sqcup\rangle$ is a cpo, F is monotonic and $\gamma(\overline{a}) \sqsubseteq F(\gamma(\overline{a}))$, they are a well-defined increasing chain and there is an ordinal ϵ such that $\mathsf{lfp}^{\sqsubseteq}_{\gamma(\overline{a})} F = F^\epsilon$.

– We let $\langle \overline{F}^\delta, \delta \in \mathbb{O}\rangle$ be the iterates of $\overline{F}$ starting from $\overline{a}$. By the hypothesis that $\overline{a}$ is a prefixpoint of the monotonic operator $\overline{F}$ on the cpo $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup}\rangle$, that they are a well-defined increasing chain and, as shown in the constructive version of Tarski's fixpoint theorem, there is an ordinal ϵ' such that $\mathsf{lfp}^{\overline{\sqsubseteq}}_{\overline{a}} \overline{F} = \overline{F}^{\epsilon'}$ [3].

– We have

[3] Once again $\langle L, \sqsubseteq, \sqcup\rangle$ and $\langle \overline{L}, \overline{\sqsubseteq}, \overline{\sqcup}\rangle$ need only be $\max(\epsilon, \epsilon')$-cpos.

– $F^0 = \gamma(\overline{a}) \sqsubseteq \gamma(\overline{a}) = \overline{F}^0$

– If $F^\delta \sqsubseteq \gamma(\overline{F}^\delta)$ by induction hypothesis, then
$F^{\delta+1} = F(F^\delta) \sqsubseteq F \circ \gamma(\overline{F}^\delta) \sqsubseteq \gamma \circ \overline{F}(\overline{F}^\delta) = \gamma(\overline{F}^{\delta+1})$

– If λ is a limit ordinal and, by induction hypothesis, $\forall \delta < \lambda : F^\delta \sqsubseteq \gamma(\overline{F}^\delta)$, then

$$F^\lambda = \bigsqcup_{\beta<\lambda} F^\beta \sqsubseteq \bigsqcup_{\beta<\lambda} \gamma(\overline{F}^\delta) \sqsubseteq \gamma(\overline{\bigsqcup_{\beta<\lambda}} \overline{F}^\delta) \quad \text{(since } \gamma \text{ is monotone)}$$

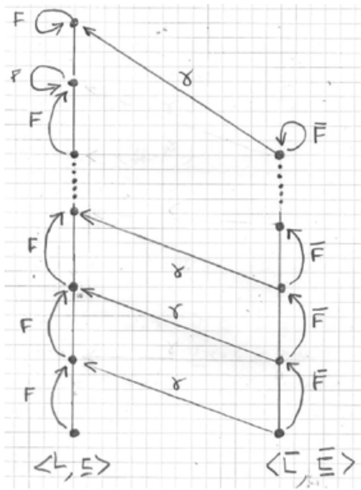
$= \gamma(\overline{F}^\lambda)$ and so $F^\lambda \sqsubseteq \gamma(\overline{F}^\lambda)$.

– By transfinite induction, $\forall \delta \in \mathbb{O} : F^\delta \sqsubseteq \gamma(\overline{F}^\delta)$.

– In conclusion, $\mathsf{lfp}^{\sqsubseteq}_{\gamma(\overline{a})} F = F^\epsilon = F^{\max(\epsilon,\epsilon')} \sqsubseteq \gamma(F^{\max(\epsilon,\epsilon')}) = \gamma(F^{\epsilon'}) = \gamma(\mathsf{lfp}^{\overline{\sqsubseteq}}_{\overline{a}} \overline{F})$.

□

Example:



γ monotonic, $F \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ \overline{F}$

This theorem is used in absence of best approximation, when a concretization function is only available (e.g. polyhedral analysis, Cousot & Halbwachs, POPL 1978).

À la Tarski, abstraction function-based fixpoint approximation

If $\langle \mathcal{D}^\natural, \sqsubseteq^\natural, \perp^\natural, \sqcup^\natural\rangle$ and $\langle \mathcal{D}^\sharp, \sqsubseteq^\sharp, \perp^\sharp, \sqcup^\sharp\rangle$ are complete lattices, $F^\natural \in \mathcal{D}^\natural \overset{m}{\longmapsto} \mathcal{D}^\natural$, $F^\sharp \in \mathcal{D}^\sharp \overset{m}{\longmapsto} \mathcal{D}^\sharp$ are monotonic and

– α is monotonic (a)

– $\forall y \in \mathcal{D}^\sharp : F^\sharp(y) \sqsubseteq^\sharp y$
$\Longrightarrow \exists x \in \mathcal{D}^\natural : \alpha(x) \sqsubseteq^\sharp y \wedge F^\natural(x) \sqsubseteq^\natural x$ (b)

then

$$\alpha(\mathsf{lfp}^{\sqsubseteq^\natural} F^\natural) \sqsubseteq^\sharp \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp$$

Proof.

$$
\begin{array}{lll}
& \alpha(\mathsf{lfp}^{\sqsubseteq^\natural} F^\natural) & \\
= & \alpha(\bigsqcap^\natural\{x \in \mathcal{D}^\natural \mid F^\natural(x) \sqsubseteq^\natural x\}) & \wr \text{Tarski} \wr \\
\sqsubseteq^\sharp & \bigsqcap^\sharp\{\alpha(x) \mid x \in \mathcal{D}^\natural \wedge F^\natural(x) \sqsubseteq^\natural x\} & \wr \alpha \text{ monotone} \wr \\
\sqsubseteq^\sharp & \bigsqcap^\sharp\{y \in \mathcal{D}^\sharp \mid \wedge F^\sharp(y) \sqsubseteq^\sharp y\} & \wr \text{by (b)} \wr \\
= & \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp & \wr \text{Tarski} \wr
\end{array}
$$

□

Sufficient conditions for iterative fixpoint computation convergence

– Given a language $\mathcal{L}$, we have seen that program properties can be defined in fixpoint form as

$$\mathsf{lfp}_{\perp[\![P]\!]}^{\sqsubseteq[\![P]\!]} F[\![P]\!]$$

where $F[\![P]\!]$ is a monotone operator on a cpo

$$\langle L[\![P]\!],\ \sqsubseteq[\![P]\!],\ \perp[\![P]\!],\ \sqcup[\![P]\!]\rangle$$

defined by structural induction on the syntactic structure of the program P

– The encoding of $F[\![P]\!]$ is essentially in two forms:
  - as a term, encoded in some data structure, together with an abstract interpreter which, when applied to the term representing $F[\![P]\!]$ and an argument $X \in L[\![P]\!]$ will return $F[\![P]\!](X)$
  - as a function, which can be directly applied to an argument $X \in L[\![P]\!]$ (this requires a functional language or code generation and is often called *abstract compilation*)

– A static analyzer is specified by an abstraction:

$$\langle L[\![P]\!],\ \sqsubseteq[\![P]\!]\rangle \underset{\alpha[\![P]\!]}{\overset{\gamma[\![P]\!]}{\leftrightarrows}} \langle \overline{L}[\![P]\!],\ \overline{\sqsubseteq}[\![P]\!]\rangle$$

and an abstract transformer:

$$\overline{F} \mathrel{\dot{\sqsupseteq}} \alpha[\![P]\!] \circ F[\![P]\!] \circ \gamma[\![P]\!]$$

which are both defined compositionally, by induction on the syntactic structure of $P \in \mathcal{L}$.

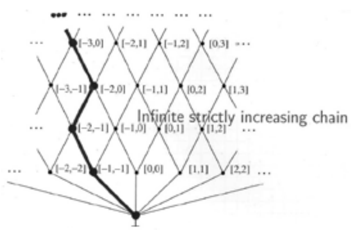
– The static analyzer has the form

```
⟨L[[P]], ⊑[[P]], ⊥[[P]], ⊔[[P]], F̄[[P]]⟩
                          := syntax_analysis(P);
X := ⊥[[P]];
repeat
    Y := F̄[[P]](X);
    stable := Y ⊑[[P]] X;
    X := Y
until stable:
diagnostic(P,X)
```


– Since $\bot[\![P]\!] \sqsubseteq[\![P]\!][\![P]\!] \overline{F}[\![P]\!](\bot[\![P]\!]$ and $\overline{F}[\![P]\!]$ is monotone, the successive values of X form a $\sqsubset[\![P]\!]$ increasing chain (but maybe for the last iterate where equality can hold). The stabilization test implies, if and when the loop exists, that $X = \overline{F}[\![P]\!](X)$. So upon termination, if ever, $X = \mathsf{lfp}^{\sqsubseteq[\![P]\!]}_{\bot[\![P]\!]} \overline{F}[\![P]\!]$ so that we can apply the soundness result.

– As far as termination is concerned, it follows that
  - The iteration terminates if the lattice $\overline{L}$ is finite
  - The iteration terminates if the lattice $\overline{L}$ satisfies the ascending chain condition (ACC).

– However, the iteration may not terminate, or terminate after a huge number of iterations. An example is the abstraction of $\wp(\mathbb{Z})$ by intervals:



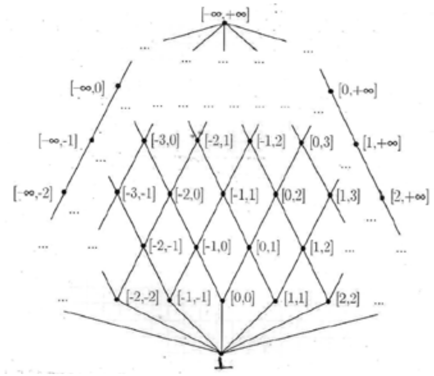
– In this case, one case choose a coarser abstraction α (in an abstract domain satisfying the ACC)

– We will later show that it is preferable to use convergence acceleration by widening/narrowing (the rôle of α is then to ensure that abstract properties have efficient computer representations, while convergence is treated otherwise, by widening/narrowing).

Iteration acceleration by extrapolation

Example of abstraction into a lattice not satisfying the ascending chain condition (ACC)

– The lattice of intervals is



– The lattice of intervals abstracts $\wp(\mathbb{Z})$:

$$\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha_{\mathbb{I}}}{\overset{\gamma_{\mathbb{I}}}{\leftrightarrows}} \langle \mathbb{I}, \sqsubseteq \rangle$$

where

- $\mathbb{I} \stackrel{\text{def}}{=} \{[a,b] \mid a,b \in \mathbb{Z} \cup \{-\infty, +\infty\} \wedge a \leq b\}$
- $\bot \sqsubseteq \bot \sqsubseteq [a,b]$
- $[a,b] \sqsubseteq [a',b']) \stackrel{\text{def}}{=} (a' \leq a \wedge b \leq b')$
- $\alpha_{\mathbb{I}}(\emptyset) \stackrel{\text{def}}{=} \bot$, when $X \neq \emptyset$, $\alpha_{\mathbb{I}}(X) \stackrel{\text{def}}{=} [\min X, \max X]$ where $\min X = -\infty$ when X has no minimum in $\mathbb{Z}$ and $\max X = +\infty$ when X has no maximum in $\mathbb{Z}$.

– The lattice of intervals provides a classical example of infinite lattice not satisfying the ascending chain condition, for which iterative fixpoint computations may not be convergent

– In practice, one can choose $-\infty$ = `min_int` and $+\infty$ = `max_int` but then the convergence, although always guaranteed is so slow that it cannot be of any practical use, but for programs with very few program variables.

Example of non-convergent iterative fixpoint computation

– Let us consider the program:

```
0: x := 1;
1: while true do
     2: x := (x + 1);
3: od
4:
```

– The states are $\Sigma \stackrel{\text{def}}{=} \{0, 1, 2, 3, 4\} \times [\texttt{min\_int}; \texttt{max\_int}]$

– The abstraction is

$$\alpha_p \in \wp(\Sigma) \mapsto \prod_{i=0}^{4} \wp([\texttt{min\_int}; \texttt{max\_int}])$$

$$\alpha_p(X) \stackrel{\text{def}}{=} \prod_{i=0}^{4} \{x \mid \langle i,\ x\rangle \in X\}$$

$$\alpha \in \wp(\Sigma) \mapsto \prod_{i=0}^{4} \{\bot\} \cup \{[\ell, h] \mid a, b \in \mathbb{Z} \wedge \texttt{min\_int} \leqslant \ell \leqslant h \leqslant \texttt{max\_int}\}$$

$$\alpha(X) \stackrel{\text{def}}{=} \prod_{i=0}^{4} (\!|\, \alpha_p(X)_i = \emptyset \;?\; \bot \;\text{¿}\; [\min \alpha_p(X)_i,\ \max \alpha_p(X)_i] \,|\!)$$

– The abstract reachable state transformer for the interval abstraction (without bounded non-modular arithmetics) can be encoded as a system of equations involving symbolic term (which can e.g. be encoded by their syntax trees)

$$\begin{cases} X0 = \{\texttt{x} \to [\texttt{min\_int}, \texttt{max\_int}]\} \\ X1 = \{\texttt{x} \to [1, 1]\} \,\dot{\sqcup}\, X3 \\ X2 = X1 \,\dot{\sqcap}\, \{\texttt{x} \to [\texttt{min\_int}, \texttt{max\_int}]\} \\ X3 = (\!|\, X2 = \bot \;?\; \bot \;\text{¿}\; \text{let } [a, b] = X2 \text{ in} \\ \qquad\qquad [\min(a+1, \texttt{max\_int}), \min(b+1, \texttt{max\_int})] \,|\!) \\ X4 = X1 \,\dot{\sqcap}\, \{\texttt{x} \to \bot\} \end{cases}$$

– A functional encoding in OCaml could be:

```
type interval = BOT | INT of (int * int);;
let less x y = match x,y with
| BOT, _ -> true
| _, BOT -> false
| INT (a,b), INT (c,d) -> (a<=c)&&(b<=d);;
let join x y = match x,y with
| BOT, _ -> y
| _, BOT -> x
| INT (a,b), INT (c,d) -> INT (min a c,max b d);;
let meet x y = match x,y with
| BOT, _ -> BOT
| _, BOT -> BOT
| INT (a,b), INT (c,d) ->
    if (b<c) or (d<a) then BOT
    else INT (max a c,min b d);;
let f (x0,x1,x2,x3,x4) =
  (INT (min_int,max_int),
   join (INT (1,1)) x3,
   meet x1 (INT (min_int,max_int)),
```

```
   (match x2 with
    | BOT -> BOT
    | INT (a,b) ->
        let a' = if a<max_int then a+1 else max_int in
          let b' = if b<max_int then b+1 else max_int in
            INT (a',b')),
   meet x1 BOT);;
let pless (x0,x1,x2,x3,x4) (x'0,x'1,x'2,x'3,x'4) =
 (less x0 x'0) && (less x1 x'1) && (less x2 x'2) && (less x3 x'3)
  && (less x4 x'4);;
let lfp leq a f =
  let rec iterate x =
    let y = f x in
      if leq y x then x
      else iterate y
  in iterate a;;
lfp pless (BOT,BOT,BOT,BOT,BOT) f;;
```


– After a few hours of computation, the result is:

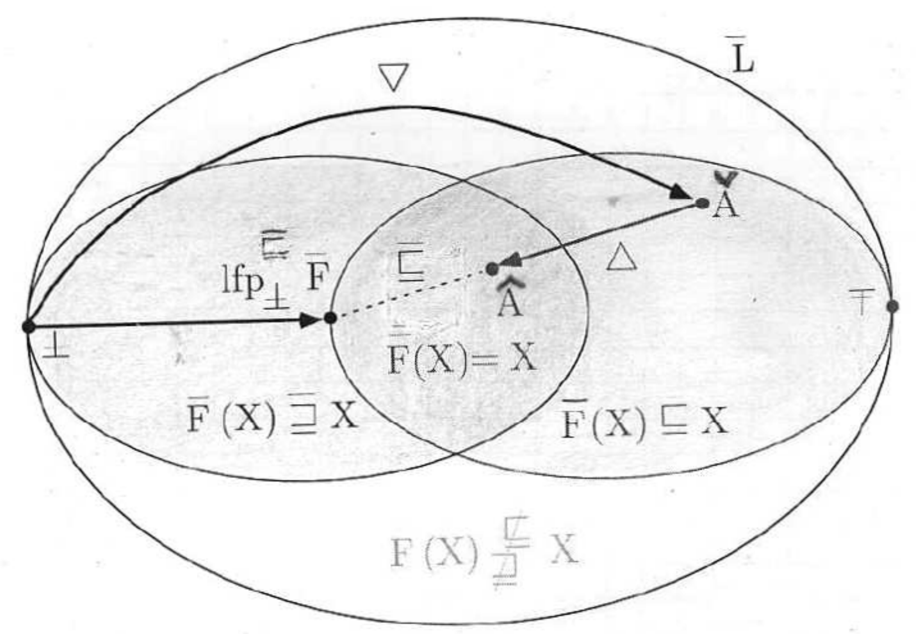
```
- : interval * interval * interval * interval * interval =
(INT (-1073741824, 1073741823), INT (1, 1073741823), INT (1, 1073741823),
 INT (2, 1073741823), BOT)
```


– The chaotic iterates from the infimum $\perp^5$ are as follows

```
X0 = { x:[-1073741824, 1073741823]
X1 = { x:[1, 1] }
X2 = { x:[1, 1] }
X3 = { x:[2, 2] }
X1 = { x:[1, 2] }
X2 = { x:[1, 2] }
X3 = { x:[2, 3] }
X1 = { x:[1, 3] }
X2 = { x:[1, 3] }
X3 = { x:[2, 4] }
X1 = { x:[1, 4] }
X2 = { x:[1, 4] }
X3 = { x:[2, 5] }
X1 = { x:[1, 5] }
X2 = { x:[1, 5] }
...
X1 = { x:[1, 1073741823] }
```


Intuition for convergence acceleration

1. Speed-up the convergence of the increasing iteration $X^0 = \perp, \ldots, X^{n+1} = F(X^n), \ldots, \breve{A}$ in order to reach a postfixpoint $\breve{A} : F(\breve{A}) \sqsubseteq \breve{A}$ so that by Tarski: $\mathsf{lfp}\, F \sqsubseteq \breve{A}$
   $\leadsto$ WIDENING ∇
2. Speed up the convergence of the decreasing iteration $Y^0 = \breve{A}, \ldots : Y^{n+1} = F(Y^n), \ldots, \hat{A}$ so as to stay <u>above</u> the least fixpoint $\mathsf{lfp}\, F \sqsubseteq \hat{A}$
   $\leadsto$ NARROWING Δ



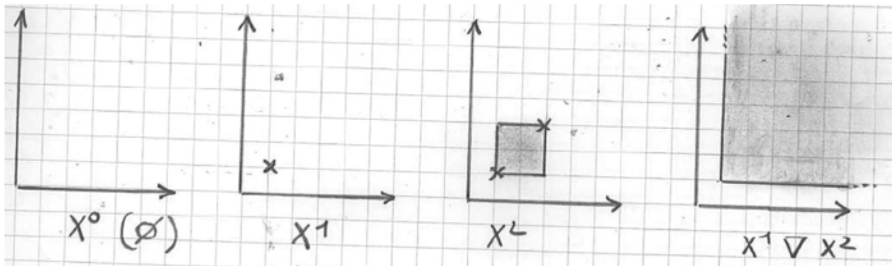
Widening

Example of widening for interval analysis

- $\overline{L} = \{\perp\} \cup \{[\ell, u] \mid \ell \in \mathbb{Z} \cup \{-\infty\} \wedge u \in \mathbb{Z} \cup \{+\infty\} \wedge \ell \leq u\}$
- The widening extrapolates unstable bounds to infinity:

$$\begin{aligned}
\perp \nabla X &= X \\
X \nabla \perp &= X \\
[\ell_0, u_0] \nabla [\ell_1, u_1] &= [(\ell_1 < \ell_0\ ?\ -\infty\ \wr\ \ell_0), \\
&\quad\ (u_1 > u_0\ ?\ +\infty\ \wr\ u_0)]
\end{aligned}$$

- Example:



- Not monotone. For example $[0, 1] \sqsubseteq [0, 2]$ but $[0, 1] \nabla [0, 2] = [0, +\infty] \not\sqsubseteq [0, 2] = [0, 2] \nabla [0, 2]$

Example of upward iteration with widening to upper-approximate a least-fixpoint by a post-fixpoint

– The analysis of the output of the following PROLOG II program:

```
program   -> init(x,1) while(x);
init(x,x) -> ;
while(x)  -> val(inf(x,100),1) out(x) line val(add(x,2),y)\\
             while(y);
```

consists in solving the equation:

$$X = \Big([1,\ 1] \sqcup (X \oplus [2,\ 2])\Big) \sqcap [-\infty,\ 99]$$

where $\emptyset \oplus I = I \oplus \emptyset = \emptyset$ and $[a,\ b] \oplus [c,\ d] = [a \dotplus c,\ b \dotplus d]$ with $-\infty \dotplus x = x \dotplus -\infty = -\infty$ and $+\infty \dotplus x = x \dotplus +\infty = +\infty$.

– Ascending abstract iteration sequence with widening:

$$\begin{aligned}
\hat{X}^0 &= \emptyset \\
\hat{X}^1 &= \hat{X}^0 \nabla \Big(([1,\ 1] \sqcup (\hat{X}^0 \oplus [2,\ 2])) \sqcap [-\infty,\ 99]\Big) \\
&= \emptyset \nabla [1,\ 1] \\
&= [1,\ 1] \\
\hat{X}^2 &= \hat{X}^1 \nabla \Big(([1,\ 1] \sqcup (\hat{X}^1 \oplus [2,\ 2])) \sqcap [-\infty,\ 99]\Big) \\
&= [1,\ 1] \nabla [1,\ 3] \\
&= [1,\ +\infty] \\
\hat{X}^3 &= \hat{X}^2 \nabla \Big(([1,\ 1] \sqcup (\hat{X}^2 \oplus [2,\ 2])) \sqcap [-\infty,\ 99]\Big) \\
&= [1,\ +\infty] \nabla [1,\ 99] \\
&= [1,\ +\infty]
\end{aligned}$$

Definition of a widening

A widening $\nabla \in P \times P \mapsto P$ on a poset $\langle P,\ \sqsubseteq\rangle$ satisfies:

– $\forall x, y \in P : x \sqsubseteq (x \nabla y) \wedge y \sqsubseteq (x \nabla y)$

– For all increasing chains $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$ the increasing chain $y^0 \stackrel{\text{def}}{=} x^0, \ldots, y^{n+1} \stackrel{\text{def}}{=} y^n \nabla x^{n+1}, \ldots$ is <u>not</u> strictly increasing.

Two different main uses:

– Approximate missing lubs.

– Convergence acceleration [4];

[4] A widening operator can be used to effectively compute an upper approximation of the least fixpoint of $\overline{F} \in \overline{L} \xrightarrow{\text{m}} \overline{L}$ starting from below when $\overline{L}$ is computer representable but does not satisfy the ascending chain condition.

Upward iteration with widening

– Let F be an operator on a poset $\langle P,\ \sqsubseteq\rangle$;

– Let $\nabla \in P \times P \mapsto P$ be a widening;

– The iteration sequence with widening ∇ for F from $\bot$ is $X^n, n \in \mathbb{N}$:

- $X^0 = \bot$
- $X^{n+1} = X^n$ if $F(X^n) \sqsubseteq (X^n)$
- $X^{n+1} = X^n \nabla F(X^n)$ if $F(X^n) \not\sqsubseteq X^n$

Correctness of the upward iteration with widening to upper-approximate a least-fixpoint by a post-fixpoint

If
- $L(\sqsubseteq, \bot, \sqcup)$ is a poset,
- $\varphi \in L \stackrel{\text{uc}}{\longmapsto} L$ and
- ∇ is a widening operator

then the increasing chain:
- $\hat{X}^0 = \bot$,
- $\hat{X}^{k+1} = \hat{X}^k$ if $\varphi(\hat{X}^k) \sqsubseteq \hat{X}^k$
- $\hat{X}^{k+1} = \hat{X}^k \nabla \varphi(\hat{X}^k)$ otherwise

for $k \in \mathbb{N}$ is stationary with limit $\hat{X}^\ell$ such that $\mathsf{lfp}\,\varphi \sqsubseteq \hat{X}^\ell$.

Proof. – $\varphi^0 = \bot \sqsubseteq \bot = \hat{X}^0 \wedge \varphi^0 \sqsubseteq \varphi^1$ [$\sqsubseteq$ reflexive, $\bot$ infimum]

– $\varphi^k \sqsubseteq \hat{X}^k \wedge \varphi^k \sqsubseteq \varphi^{k+1}$ [induction hypothesis]

$\Longrightarrow \varphi^{k+1} = \varphi(\varphi^k) \sqsubseteq \varphi(\hat{X}^k) \wedge \varphi^{k+1} \sqsubseteq \varphi^{k+2}$ [monotony]

- if $\varphi(\hat{X}^k) \sqsubseteq \hat{X}^k$ then $\hat{X}^{k+1} = \hat{X}^k$

  $\Longrightarrow \varphi^{k+1} \sqsubseteq \hat{X}^{k+1}$ [transitivity]

- else $\hat{X}^{k+1} = \hat{X}^k \nabla \varphi(\hat{X}^k)$

  $\Longrightarrow \hat{X}^k \sqsubseteq \hat{X}^{k+1} \wedge \varphi^{k+1} \sqsubseteq \hat{X}^{k+1}$ [(b)]

$\Longrightarrow$ the chain $\hat{X}^k$, $k \in \mathbb{N}$ is increasing and [by induction]

(1) $\forall k \in \mathbb{N} : \varphi^k \sqsubseteq \hat{X}^k$.

$\Longrightarrow$ the chain $\varphi(\hat{X}^k)$, $k \in \mathbb{N}$ is increasing [monotony]

$\Longrightarrow$ the chain $\hat{X}^k$, $k \in \mathbb{N}$ is stationary. [(c)]

– For the limit $\hat{X}^\ell$ where $\ell \in \mathbb{N}$, we have:

(2) - $\forall k \leq \ell$: $\hat{X}^k \sqsubseteq \hat{X}^\ell$ [increasing chain]

- $m \geq \ell \wedge \hat{X}^m = \hat{X}^\ell$ [induction hypothesis]

  $\Longrightarrow \hat{X}^{m+1} = \hat{X}^m = \hat{X}^\ell$ [if $\varphi(\hat{X}^m) \sqsubseteq \hat{X}^m$]

  or $\Longrightarrow \hat{X}^{m+1} = \hat{X}^m \nabla \varphi(\hat{X}^m)$ [otherwise]

  $= \hat{X}^\ell \nabla \varphi(\hat{X}^\ell) = \hat{X}^\ell$

  $\Longrightarrow \hat{X}^{m+1} = \hat{X}^\ell$ [by cases]

(3) $\Longrightarrow \forall m \geq \ell : \hat{X}^m = \hat{X}^\ell$ [by induction]

$\Longrightarrow \forall k \in \mathbb{N} : \varphi^k \sqsubseteq \hat{X}^k \sqsubseteq \hat{X}^\ell$ [(1) (2) (3)]

$\Longrightarrow \mathsf{lfp}\,\varphi = \bigsqcup_{k\in\mathbb{N}} \varphi^k \sqsubseteq \hat{X}^\ell$ [Tarski constructive and lubs]

□

The generalization to a monotonic $\varphi \in L \stackrel{\text{m}}{\longmapsto} L$ is straightforward.

In summary:

- Any iteration sequence with widening is increasing and stationary after finitely many iteration steps;
- Its limit F^{∇} is a post-fixpoint of F, whence an upper-approximation of the least fixpoint $\mathsf{lfp}^{\sqsubseteq} F$ [5]:

$$\mathsf{lfp}^{\sqsubseteq} F \sqsubseteq F^{\nabla}$$

[5] if $\mathsf{lfp}^{\sqsubseteq} F$ does exist e.g. if $\langle P, \sqsubseteq, \bot, \cup \rangle$ is a cpo.

Example of convergence acceleration of an upward iterative fixpoint computation by widening

- Program:

```
0: x := 1;
2: while (x < 1000) do
     3: x := (x + 1);
4: od {(x >= 1000))}
6:
```


- Forward abstract equations for interval analysis with widening:

$$\begin{cases} X0 = \{\mathtt{x} \to [\mathtt{min\_int}, \mathtt{max\_int}]\} \\ X2 = \{\mathtt{x} \to [1, 1]\} \mathbin{\dot{\sqcup}} X4 \\ X3 = X2 \mathbin{\dot{\sqcap}} \{\mathtt{x} \to [\mathtt{min\_int}, 999]\} \\ X4 = (\!| X3 = \bot \mathrel{?} \bot \mathrel{⁝} \text{let } [a, b] = X3 \text{ in} \\ \qquad\qquad [\min(a + 1, \mathtt{max\_int}), \min(b + 1, \mathtt{max\_int})] |\!) \\ X6 = X2 \mathbin{\dot{\sqcap}} \{\mathtt{x} \to [1000, \mathtt{max\_int}]\} \end{cases}$$

- Iteration with widening from X0 = X2 = X3 = X4 = X6 = {x → ⊥}:

```
X2 = { x:_|_ }
widening at 2 by { x:[1, 1] }
X2 = { x:[1, 1] }
widening at 3 by { x:[1, 1] }
X3 = { x:[1, 1] }
widening at 4 by { x:[2, 2] }
X4 = { x:[2, 2] }
widening at 2 by { x:[1, 2] }
X2 = { x:[1, +oo] }
widening at 3 by { x:[1, 999] }
X3 = { x:[1, +oo] }
widening at 4 by { x:[2, +oo] }
X4 = { x:[2, +oo] }
widening at 6 by { x:[1000, +oo] }
X6 = { x:[1000, +oo] }
stable
```

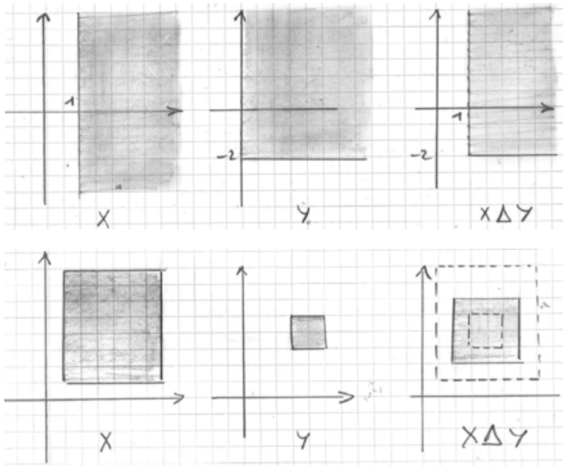
Narrowing

Example of narrowing for interval analysis

– The narrowing improves infinite bounds only:

$$\bot \,\Delta\, X = \bot$$
$$[\ell_0,\, u_0] \,\Delta\, [\ell_1,\, u_1] = [(\ell_0 = -\infty \;?\; \ell_1 \;¿\; \ell_0),\; (u_0 = +\infty \;?\; u_1 \;¿\; u_0)]$$

– Other examples of narrowings:



Example of downward iteration with narrowing to improve a post-fixpoint approximation of a (least) fixpoint

– Equation (contn'd):

$$X = \Big([1,\, 1] \sqcup (X \oplus [2,\, 2])\Big) \sqcap [-\infty,\, 99]$$

where $\emptyset \oplus I = I \oplus \emptyset = \emptyset$ and $[a,\, b] \oplus [c,\, d] = [a \dotplus c,\, b \dotplus d]$ with $-\infty \dotplus x = x \dotplus -\infty = -\infty$ and $+\infty \dotplus x = x \dotplus +\infty = +\infty$.

– Descending abstract iteration sequence with narrowing starting from $\widetilde{X}^3 = [1, +\infty]$:

$$
\begin{aligned}
\check{X}^0 &= \hat{X}^3 \\
&= [1,\ +\infty] \\
\check{X}^1 &= \check{X}^0 \,\Delta\, \Big(([1,\ 1] \sqcup (\check{X}^0 \oplus [2,\ 2])) \sqcap [-\infty,\ 99]\Big) \\
&= [1,\ +\infty] \,\Delta\, [1,\ 99] \\
&= [1,\ 99] \\
\check{X}^2 &= \check{X}^1 \,\Delta\, \Big(([1,\ 1] \sqcup (\check{X}^1 \oplus [2,\ 2])) \sqcap [-\infty,\ 99]\Big) \\
&= [1,\ 99] \,\Delta\, [1,\ 99] \\
&= [1,\ 99]
\end{aligned}
$$

The analysis time does not depend upon the number of iterations in the while-loop.

Definition of the narrowing

– Since we have got a postfixpoint F^{∇} of $F \in P \mapsto P$, its iterates $F^n(F^{\nabla})$ are all upper approximations of $\mathsf{lfp}\ F$.

– To accelerate convergence of this decreasing chain, we use a narrowing $\nabla \in P \times P \mapsto P$ on the poset $\langle P, \sqsubseteq\rangle$ satisfying:

- $\forall x, y \in P : y \sqsubseteq x \Longrightarrow y \sqsubseteq x \,\Delta\, y \sqsubseteq x$
- For all decreasing chains $x^0 \sqsupseteq x^1 \sqsupseteq \ldots$ the decreasing chain $y^0 \stackrel{\text{def}}{=} x^0, \ldots, y^{n+1} \stackrel{\text{def}}{=} y^n \,\Delta\, x^{n+1}, \ldots$ is not strictly decreasing.

Decreasing Iteration Sequence with Narrowing

– Let F be a monotonic operator on a poset $\langle P, \sqsubseteq\rangle$;

– Let $\Delta \in P \times P \mapsto P$ be a narrowing;

– The iteration sequence with narrowing Δ for F from the postfixpoint P[6] is $Y^n, n \in \mathbb{N}$:

- $Y^0 = P$
- $Y^{n+1} = Y^n \quad$ if $F(Y^n) = Y^n$
- $Y^{n+1} = Y^n \,\Delta\, F(Y^n) \quad$ if $F(Y^n) \neq Y^n$

[6] $F(P) \sqsubseteq P$.

Correctness of the downward iteration with narrowing to improve a post-fixpoint approximation of a (least) fixpoint

If

- $L(\sqsubseteq)$ is a poset,
- $\varphi \in L \stackrel{\text{m}}{\longmapsto} L$,
- $\Delta \in L \times L \mapsto L$ is a narrowing operator and
- $\varphi(x) = x \sqsubseteq y,\ \varphi(y) \sqsubseteq y$,

then the decreasing chain:

- $\check{X}^0 = y$,
- $\check{X}^{k+1} = \check{X}^k \,\Delta\, \varphi(\check{X}^k)$

for $k \in \mathbb{N}$ is stationary with limit $\check{X}^\ell, \ell \in \mathbb{N}$ such that $x \sqsubseteq \check{X}^\ell \sqsubseteq y$.

PROOF. – $x \sqsubseteq \check{X}^0$ [hypothesis and transitivity]

– $x \sqsubseteq \check{X}^k$ [induction hypothesis]

$\Longrightarrow x = \varphi(x) \sqsubseteq \varphi(\check{X}^k)$ [monotony]

$\Longrightarrow x \sqsubseteq \check{X}^{k+1} = \check{X}^k \Delta \varphi(\check{X}^k) \sqsubseteq \check{X}^k$ [(e) and (f)]

$\Longrightarrow \forall k \in \mathbb{N} : x \sqsubseteq \check{X}^k$ and [by induction]

the chain $\check{X}^k$, $k \in \mathbb{N}$ is decreasing for $\sqsubseteq$

$\Longrightarrow$ the chain $\varphi(\check{X}^k)$, $k \in \mathbb{N}$ is decreasing for $\sqsubseteq$ [monotony]

$\Longrightarrow \check{X}^k$, $k \in \mathbb{N}$ has a limit $\check{X}^\ell$ [(g)]

$\Longrightarrow x \sqsubseteq \check{X}^\ell \sqsubseteq \check{X}^0 = y$.

□

In summary:

– Any iteration sequence with narrowing starting from a postfixpoint P of F[7] is decreasing and stationary after finitely many iteration steps;

– if $\mathsf{lfp}^{\sqsubseteq} F$ does exist[8] and $\mathsf{lfp}^{\sqsubseteq} F \sqsubseteq P$ then its limit F^{Δ} is a fixpoint of F, whence an upper-approximation of the least fixpoint $\mathsf{lfp}^{\sqsubseteq} F$:

$$\mathsf{lfp}^{\sqsubseteq} F \sqsubseteq F^{\Delta} \sqsubseteq P$$

– The downward iteration sequence can jump over <u>no</u> fixpoint (hence cannot jump over the [unknown] least fixpoint), which ensures that we have an approximation from above.

[7] $F(P) \sqsubseteq P$

[8] e.g. if $\langle P, \sqsubseteq, \bot, \cup\rangle$ is a cpo.

Example of convergence acceleration of a downward iteration with narrowing to improve a post-fixpoint approximation of a (least) fixpoint

– Program:

```
0: x := 1;
2: while (x < 1000) do
     3: x := (x + 1);
4: od {(x >= 1000))}
6:
```


– Forward abstract equations for interval analysis with widening:

$$\begin{cases} X0 = \{\mathtt{x} \to [\mathtt{min\_int}, \mathtt{max\_int}]\} \\ X2 = \{\mathtt{x} \to [1,1]\} \dot{\sqcup} X4 \\ X3 = X2 \dot{\sqcap} \{\mathtt{x} \to [\mathtt{min\_int}, 999]\} \\ X4 = (X3 = \bot \mathrel{?} \bot \mathrel{\text{¦}} \text{let } [a,b] = X3 \text{ in} \\ \qquad\qquad [\min(a+1, \mathtt{max\_int}), \min(b+1, \mathtt{max\_int})]) \\ X6 = X2 \dot{\sqcap} \{\mathtt{x} \to [1000, \mathtt{max\_int}]\} \end{cases}$$

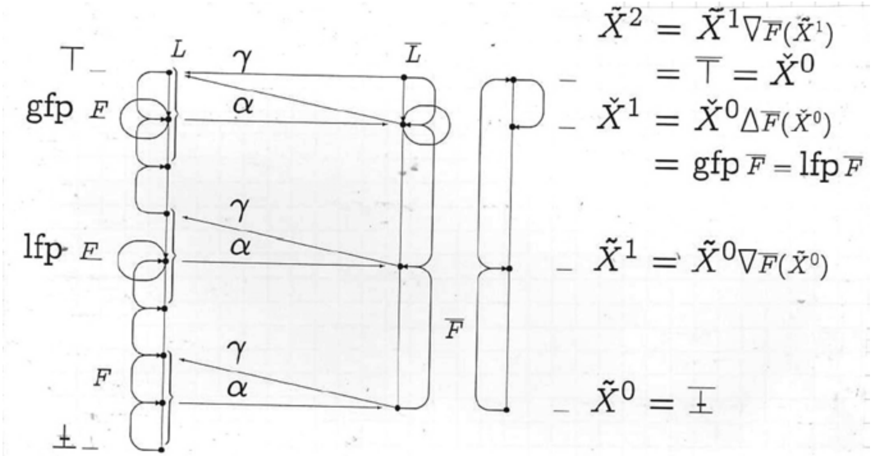
```
iterations with narrowing from:
X0 = { x:_O_ }
X2 = { x:[1,+oo] }
X3 = { x:[1,+oo] }
X4 = { x:[2,+oo] }
X6 = { x:[1000,+oo] }
-
narrowing at 0 by { x:[-oo,+oo]}
X0 = { x:[-oo,+oo]}
narrowing at 2 by { x:[1,+oo] }
X2 = { x:[1,+oo] }
narrowing at 3 by { x:[1,999] }
X3 = { x:[1,999] }
narrowing at 4 by { x:[2,1000] }
X4 = { x:[2,1000] }
narrowing at 6 by { x:[1000,+oo] }
X6 = { x:[1000,+oo] }
narrowing at 0 by { x:[-oo,+oo] }
X0 = { x:[-oo,+oo]}
narrowing at 2 by { x:[1,1000] }
X2 = { x:[1,1000] }
narrowing at 3 by { x:[1,999] }
X3 = { x:[1,999] }
narrowing at 4 by { x:[2,1000] }
X4 = { x:[2,1000] }
narrowing at 6 by { x:[1000,1000] }
X6 = { x:[1000,1000] }
stable
```

– Obviously narrowing at each program point can be replaced by a narrowing at loop heads (see later). Was the same for widenings.

Static Analysis with Widening/Narrowing

Iteration convergence acceleration

– Intuition:



– A non-trivial example of automatic interval analysis with widening/narrowing:

```
function F(X : integer) : integer;
begin
   if X > 100 then begin
      F := X - 10
      { X:101..maxint & F:91..maxint - 10 }
   end else begin
      F := F(F(X + 11))
      { X:minint..100 & F = 91 }
   end;
end;
```

(simple ideas can be effective but in general more refined widenings should be used, as shown later).

On fixpoint approximation using widening/narrowing operators

- The approximation is done a priori, once for all ($L \stackrel{\gamma}{\underset{\alpha}{\leftrightarrows}} \overline{L}$, ∇ and Δ).
- The approximation α may be precise while ∇ may be very rough.
- Usefulness of the approximation is shown by experience (precision/cost can be tuned with ∇).
- The approximation is applied at each iteration step for $\overline{F}$.
- The approximation depends upon the iterates.

- $\overline{L}$ need not satisfy the ascending chain condition (since ∇ will be used to enforce convergence).

Schema of a static program analyzer with widening/narrowing

$\langle \overline{\bot}, \overline{F} \rangle$:= syntactic_analysis(Program);
%% $\alpha(\bot) \sqsubseteq \overline{\bot} \wedge \alpha \circ F \circ \gamma \sqsubseteq \overline{F}$
$X := \overline{\bot}$;
repeat
$\quad Y := X$;
$\quad X := \overline{F}(X)$;
$\quad$ **if** $X \sqsubseteq Y$ **then** $C :=$ true
$\quad$ **else** $C :=$ false; $X := Y \mathbin{\nabla} X$ **fi**
until C;
%% $\mathsf{lfp}^{\overline{\bot}}\, \overline{F} \sqsubseteq X = \overline{F}(Y) \sqsubseteq Y \wedge \mathsf{lfp}^{\bot}\, F \sqsubseteq \gamma(Y)$

%% $\mathsf{lfp}^{\overline{\bot}}\, \overline{F} \sqsubseteq X = \overline{F}(Y) \sqsubseteq Y$
while $X \neq Y$ **do**
$\quad Y := Y \mathbin{\Delta} X$;
$\quad X := \overline{F}(Y)$
od;
%% $\mathsf{lfp}^{\overline{\bot}}\, \overline{F} \sqsubseteq \overline{F}(X) = X \wedge \mathsf{lfp}^{\bot}\, F \sqsubseteq \gamma(X)$

In practice, chaotic or asynchronous iterations (with memory).

Galois-connection based static program analyzer with widening/narrowing

- The Galois connection approach is the basic method of abstract interpretation.
- With its variants (e.g. concretization function only in absence of best approximation), its always applicable to a poset with ACC;
- However, combination with the widening/narrowing is the key to success:
  - Rich domain of information (whence not satisfying the ACC),
  - Convergence acceleration.
- In practice, a much better compromise than just weakening the expressiveness of the abstract domain using a coarser Galois connection.

Properties of Widening/Narrowing

Widening/narrowing are not dual; Dual widening/narrowing

- The iteration with widening starts from below the least fixpoint and stabilizes above to a postfixpoint;
- The iteration with narrowing starts from above the least fixpoint and stabilizes above;
- The iteration with dual widening starts from above the greatest fixpoint and stabilizes below to a prefixpoint;
- The iteration with dual narrowing starts from below the greatest fixpoint and stabilizes below;

	Iteration starts from	Iteration stabilizes
Widening ∇	below	above
Narrowing Δ	above	above
Dual widening $\widetilde{\nabla}$	above	below
Dual narrowing $\widetilde{\Delta}$	below	below

Whence that's four different notions.

An example of static analysis of a simple program for automatic determination of interval invariant by fixpoint approximation with convergence acceleration by widening/narrowing

```
program P;
    var I : integer;
begin
{1:}
    I := 1;
{2:}
    while { I ∈ X } I <= 100 do begin
{3:}
        I := I + 2;
{4:}
    end;
{5:} { I ∈ Y }
end.
```


– Interval equations:
  - $X = [1, 1] \cup ((X \cap [-\infty, 100]) \mp [2, 2])$
  - $Y = X \cap [101, +\infty]$

– Upwards iteration from the infimum without widening

$$
\begin{array}{llll}
X^0 = \bot & & Y^0 = \bot \\
X^1 = [1, 1] & & Y^1 = \bot \\
X^2 = [1, 3] & & Y^2 = \bot \\
X^3 = [1, 5] & & Y^3 = \bot \\
\ldots = \ldots & & \ldots = \ldots \\
X^{50} = [1, 99] & & Y^{50} = \bot \\
X^{51} = [1, 101] & & Y^{51} = [101, 101] \\
X^{52} = [1, 101] & & Y^{52} = [101, 101]
\end{array}
$$

Convergence could have been very slow (even impossible without the test `I <= 100` when using bignums)!

– Upwards iteration from the infimum with widening

$$
\begin{array}{l}
\hat{X}^0 = \bot \\
\hat{X}^1 = \hat{X}^0 \nabla [1, 1] = [1, 1] \\
\hat{X}^2 = \hat{X}^1 \nabla [1, 3] = [1, +\infty] \\
\hat{X}^3 = \hat{X}^2 \nabla [1, 102] = [1, +\infty] \\
\hat{Y}^3 = [101, +\infty]
\end{array}
$$

Convergence is accelerated hence the loss of precision!

– Downward iteration with narrowing

$$
\begin{array}{ll}
\check{X}^0 = [1, +\infty] & \check{Y}^0 = [101, +\infty] \\
\check{X}^1 = \check{X}^0 \,\Delta\, [1, 102] = [1, 102] & \\
\check{X}^2 = \check{X}^1 \,\Delta\, [1, 102] = [1, 102] & \\
 & \check{Y}^3 = [101, 102]
\end{array}
$$

– The narrowing is not always able to recapture the information lost by the widening

– It's therefore better not to loose too much information by widening in the first upward iteration

A parameterized meta-example of interval invariant by fixpoint approximation with convergence acceleration by widening/narrowing

– The analyzer will behave in exactly the same way for all programs of the form

```
program P;
    var I : integer;
begin
{1:}
    I := 1;
{2:}
    while { I ∈ X } I <= n do begin
{3:}
        I := I + 2;
{4:}
    end;
{5:} { I ∈ Y }
end.
```


where n is a mathematical variable denoting any program constant, $n \geq 1$. By instantiating n to all possible naturals ≥ 1, one gets an infinite family of programs, which are similar up to n and have therefore similar analyzes.

– The fixpoint interval equations are all of the same form:
  - $X = [1,\ 1] \cup ((X \cap [-\infty,\ n]) \mp [2,\ 2]) \quad X \in [1,\ n+1]$
  - $Y = X \cap [n+1,\ +\infty] \qquad Y \in [n+1,\ +\infty]$

– The upward iteration with widening is now (in parametric form):

$$
\begin{aligned}
\hat{X}^0 &= \bot\\
\hat{X}^1 &= \hat{X}^0 \,\nabla\, ([1,\ 1] \cup ((\hat{X}^0 \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= \bot \,\nabla\, [1,\ 1] \cup \bot\\
&= [1,\ 1]\\
\hat{X}^2 &= \hat{X}^1 \,\nabla\, ([1,\ 1] \cup ((\hat{X}^1 \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ 1] \,\nabla\, ([1,\ 1] \cup (([1,\ 1] \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ 1] \,\nabla\, [1,\ 3]\\
&= [1,\ +\infty]\\
\hat{X}^3 &= \hat{X}^2 \,\nabla\, ([1,\ 1] \cup ((\hat{X}^2 \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ +\infty] \,\nabla\, ([1,\ 1] \cup (([1,\ +\infty] \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ +\infty] \,\nabla\, ([1,\ 1] \cup ([3,\ n+2])\\
&= [1,\ +\infty] \,\nabla\, ([1, n+2])\\
&= [1,\ +\infty]
\end{aligned}
$$

– The downward iteration sequence from $[1,\ +\infty]$ with narrowing will now be as follows (always in parameterized form, to be instantiated for any particular value of n):

$$
\begin{aligned}
\check{X}^0 &= [1,\ +\infty]\\
\check{X}^1 &= \check{X}^0 \,\Delta\, ([1,\ 1] \cup ((\check{X}^0 \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ +\infty] \,\Delta\, ([1,\ 1] \cup (([1,\ +\infty] \cap [-\infty,\ n]) \mp [2,\ 2]))\\
&= [1,\ +\infty] \,\Delta\, ([1,\ 1] \cup (([1,\ n]) \mp [2,\ 2]))\\
&= [1,\ +\infty] \,\Delta\, ([1,\ 1] \cup [2,\ n+2])\\
&= [1,\ +\infty] \,\Delta\, ([1, n+2])\\
&= [1, n+2])\\
\check{X}^2 &= \check{X}^1 \,\Delta\, [1,\ n+2]\\
&= [1,\ n+2]
\end{aligned}
$$

$$
\begin{aligned}
\check{Y}^2 &= \check{X}^2 \cap [n+1,\ +\infty] \\
&= [1,\ n+2] \cap [n+1,\ +\infty] \\
&= [n+1,\ n+2]
\end{aligned}
$$

– This proves that for all programs in the family (parameterized by n), the analysis with widening/narrowing will always discover the interval invariant

$$
\begin{cases} X = [1,\ n+2] \\ Y = [n+1,\ n+2] \end{cases}
$$

for the given n corresponding to each particular program in the family.

Finitary nature of static analysis with widening/narrowing

THEOREM. Given any specific program, and given specific infinite abstract domain together with a specific widening, it is possible to find a finite lattice and a Galois connection which will produce exactly the same analysis results for that given program. ∎

PROOF. – Assume that we are given a program P and that the problem is to overapproximate $\mathsf{lfp}_{\bot}^{\sqsubseteq} F$ where F is a concrete monotonic transformer $F \in L \overset{m}{\longmapsto} L$ on the cpo $\langle L,\ \sqsubseteq,\ \bot,\ \top,\ \sqcup\rangle$. We assume L to contain a supremum $\top$ [9]

[9] to be able to express "I don't know" in the concrete.

– The analyzer makes use of an abstract domain $\langle \overline{L},\ \overline{\sqsubseteq}\rangle$ such that $\langle L,\ \sqsubseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftarrows\!\!\twoheadrightarrow}} \langle \overline{L},\ \overline{\sqsubseteq}\rangle$, a monotonic abstract transformer $\overline{F}\ \dot{\overline{\sqsupseteq}}\ \alpha \circ F \circ \gamma$ and a widening $\overline{\nabla}$.

– Because α is surjective [10], $\langle \overline{L},\ \overline{\sqsubseteq},\ \overline{\bot},\ \overline{\top},\ \overline{\sqcup}\rangle$ is indeed a cpo with supremum $\overline{\top} = \alpha(\top)$.

– The analysis computes iterates $y^0 = \alpha(\bot),\ \ldots,\ y^{n+1} = y^n\ \overline{\nabla}\ \overline{F}(y^n), \ldots, y^\ell$ where the limit y^ℓ is a postfixpoint $\overline{F}(y^\ell)\ \overline{\sqsubseteq}\ y^\ell$

– Let us define the abstract domain $\overline{\overline{L}} = \{y^0, \ldots, y^n, \ldots, y^\ell, \overline{\top}\}$ with ordering $\overline{\overline{\sqsubseteq}}$ which is $\overline{\sqsubseteq}$ restricted to $\overline{\overline{L}}$.

– Because the iterates are a finite increasing chain and $\overline{\top}\}$ is the supremum, $\langle \overline{\overline{L}},\ \overline{\overline{\sqsubseteq}}\rangle$ is a finite chain whence a complete lattice.

– Let us define the abstraction

$$
\begin{aligned}
\overline{\overline{\alpha}} &\in L \mapsto \overline{\overline{L}} \\
\overline{\overline{\alpha}}(x) &\stackrel{\text{def}}{=} \overline{\overline{\bigsqcap}}\{y \in \overline{\overline{L}} \mid \alpha(x)\ \overline{\sqsubseteq}\ y\}\ ^{11}
\end{aligned}
$$

[10] Otherwise we choose $\overline{L} = \alpha(L)$.

[11] So that $\alpha(x) = \overline{\top}$ if x is not comparable to any of the iterates y^i, $i = 0, \ldots, \ell$.

and the concretization

$$
\begin{aligned}
\overline{\overline{\alpha}} &\in \overline{\overline{L}} \mapsto L \\
\overline{\overline{\gamma}} &\stackrel{\text{def}}{=} \gamma
\end{aligned}
$$

– We have a Galois connection $\langle L,\ \sqsubseteq\rangle \underset{\overline{\overline{\alpha}}}{\overset{\overline{\overline{\gamma}}}{\leftarrows\!\!\twoheadrightarrow}} \langle \overline{\overline{L}},\ \overline{\overline{\sqsubseteq}}\rangle$.

PROOF.

$$
\begin{aligned}
& \overline{\overline{\alpha}}(x)\ \overline{\overline{\sqsubseteq}}\ y \\
\Longleftrightarrow\ & \overline{\overline{\alpha}}(x)\ \overline{\sqsubseteq}\ y && \text{(def. } \overline{\overline{\sqsubseteq}}\text{)} \\
\Longleftrightarrow\ & \overline{\overline{\bigsqcap}}\{y \in \overline{\overline{L}} \mid \alpha(x)\ \overline{\sqsubseteq}\ y\}\ \overline{\sqsubseteq}\ y && \text{(def. } \overline{\overline{\alpha}}\text{)} \\
\Longrightarrow\ & && \text{(We have } \alpha(x)\ \overline{\sqsubseteq}\ \overline{\top} \in \overline{\overline{L}} \text{ so, since } \overline{\overline{L}} \text{is a finite strictly decreasing} \\
& && \text{chain, there is a smallest } y^n \in \overline{\overline{L}} : \alpha(x)\ \overline{\sqsubseteq}\ y^n\text{, whence } \alpha(x)\ \overline{\sqsubseteq}\ y' \\
& && \text{implies } y^n\ \overline{\sqsubseteq}\ y' \text{ so } y^n = \overline{\overline{\bigsqcap}}\{y \in \overline{\overline{L}} \mid \alpha(x)\ \overline{\sqsubseteq}\ y\}\ \overline{\sqsubseteq}\ y\text{. It follows that:)} \\
& \alpha(x)\ \overline{\sqsubseteq}\ y^n\ \overline{\overline{\sqsubseteq}}\ y \\
\Longrightarrow\ & x \sqsubseteq \gamma(y)
\end{aligned}
$$

$$\Longrightarrow x \sqsubseteq \overline{\overline{\gamma}}(y) \qquad \wr\text{since } \gamma = \overline{\overline{\gamma}}\wr$$

Conversely

$$x \sqsubseteq \overline{\overline{\gamma}}(y)$$
$$\Longrightarrow x \sqsubseteq \gamma(y) \qquad \wr\text{since } \gamma = \overline{\overline{\gamma}}\wr$$
$$\Longrightarrow \alpha(x) \overline{\sqsubseteq} y$$
$$\Longrightarrow \overline{\overline{\bigsqcap}}\{y' \in \overline{\overline{L}} \mid \alpha(x) \overline{\sqsubseteq} y'\} \overline{\sqsubseteq} y$$
$$\Longrightarrow \overline{\overline{\alpha}}(x) \overline{\sqsubseteq} y$$
$$\Longrightarrow \overline{\overline{\alpha}}(x) \overline{\overline{\sqsubseteq}} y$$

□

– Let us define:

$$\overline{\overline{F}} \in \overline{\overline{L}} \mapsto \overline{\overline{L}}$$
$$\overline{\overline{F}} \stackrel{\text{def}}{=} \lambda y \cdot (\!| \, y = \top \; ? \; \top \; \| \; y = y^{\ell} \; ? \; \overline{F}(y^{\ell}) \; ¿ \; y \, \nabla \, \overline{F}(y) \, |\!)$$

– We have $\overline{\overline{F}} \mathrel{\dot{\sqsupseteq}} \overline{F} \mathrel{\dot{\sqsupseteq}} \overline{\overline{\alpha}} \circ F \circ \overline{\overline{\gamma}}$

PROOF. We proceed pointwise on $\overline{\overline{L}} \subseteq \overline{L}$. We already know that $\overline{F} \mathrel{\dot{\sqsupseteq}} \overline{\overline{\alpha}} \circ F \circ \overline{\overline{\gamma}}$.

- This is obvious for $\top$ since $\overline{\overline{F}}(\top) = \top \sqsupseteq \overline{F}(\top)$ since $\top$ is the supremum of $\overline{L}$
- This holds for y^{ℓ} since $\overline{\overline{F}}(y^{\ell}) = \overline{F}(y^{\ell}) \sqsupseteq \overline{F}(y^{\ell})$ by reflexivity.
- For the other elements $y \in \overline{\overline{L}} \setminus \{\top, y^{\ell}\}$, we have $\overline{\overline{F}}(y) = y \, \nabla \, \overline{F}(y) \sqsupseteq \overline{F}(y)$ by def. ∇ which is an upper bound.
- Observe that the iterates of $\overline{\overline{F}}$ from $\alpha(\bot)$ are exactly $y^0 = \alpha(\bot)$, ..., $y^n = \overline{\overline{F}}(y^{n-1}) = y^{n-1} \, \nabla \, \overline{F}(y^{n-1})$, ..., y^{ℓ} since $\overline{\overline{F}}(y^{\ell}) = \overline{F}(y^{\ell}) \sqsubseteq y^{\ell}$, which is the convergence condition.
- These iterates are therefore convergent (despite the fact that the widening ∇ hence $\overline{\overline{F}}$ is not monotone since $\overline{\overline{F}}$ is extensive, but for y^{ℓ}, which is the convergence point).

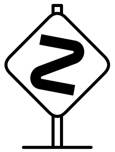
- In conclusion, the analysis of the given program can be done in the finite (complete) lattice $\langle \overline{\overline{L}}, \overline{\overline{\sqsubseteq}}, \overline{\overline{\bot}}, \overline{\overline{\sqcup}} \rangle$ by computing the limit (y^{ℓ}) of the finitely convergent iterates of $\overline{\overline{F}} \mathrel{\dot{\sqsupseteq}} \overline{\overline{\alpha}} \circ F \circ \overline{\overline{\gamma}}$.

□

□

An incorrect common believe about the uselessness of widenings

Because of this theorem, some (a.o. [1]) conclude that:



> The widening approach to program static analysis is useless since it is always possible to perform an iterative static analysis using a finite abstract domain.

This is ERRONEOUS [2]

Reference

[1] R.B. Kieburtz and M. Napierala. Abstract semantics. In S. Abramsky and C. Hankin, eds., Abstract Interpretation of Declarative Languages, chapter 7, pp. 143–180. Ellis Horwood, Chichester, U.K., 1987.

[2] P. Cousot and R. Cousot. Comparing the Galois Connection and Widening/Narrowing Approaches to Abstract Interpretation, invited paper. In M. Bruynooghe and M. Wirsing, eds., *Programming Language Implementation and Logic Programming, Proc. 4th In. Symp., PLILP'92*, Leuven, Belgium, 13–17 Aug. 1992, LNCS 631, p. 269–295. Springer-Verlag, 1992.

Proof that the common believe about the uselessness of widenings is erroneous

– This is due to the confusion between analyzing a given program, as opposed to any program in a given programming language

– We exhibit a counter-example, using interval analysis, showing that infinitely many $[0, n], n \geq 0$ are needed

– For any given $n = 0, 1, 2, \ldots$, we have seen that the interval analysis with widening will produce the following analysis (given as comments between $\{\ldots\}$):


```
program P;
    var I : integer;
begin
{1:}
      I := 1;
{2:}
      while { I ∈ X } I <= 100 do begin
{3:}
         I := I + 2;
{4:}
      end;
{5:} { I ∈ Y }
end.
```

– So when considering all programs $P(n)$, for all $n \geq 0$, we have to have all necessary inductive invariants $[0, n + 1]$, $n \geq 0$ (otherwise the analysis can only be less precise if this invariant is not expressible)

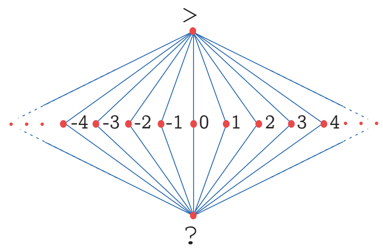
– So the abstract domain with which the static analysis can produce this result for all $P(n)$, $n \geq 0$ must contain an infinite strictly increasing chain $[0, 1] \sqsubseteq [0, 2] \sqsubseteq \ldots \sqsubseteq [0, n] \sqsubseteq \ldots$

– Analyzing iteratively a program like `while true do I := I + 2; end` will definitely require a widening to converge

– Another hope would be to guess the constants n, $\ldots$ by a simple syntactic inspection of the program text (by "simple" we exclude a static analysis with widening and similar sophisticated analyzes!)

– However practice show that this is extremely difficult

– A first example is Kildall's constant propagation using the lattice:



for which an equally precise analysis has to guess all necessary constants, including those not appearing explicitly in the program text

– A second example, using interval analysis

```
program Variant_of_function_91_of_McCarthy;
   var X, Y : integer;
   function F(X : integer) : integer;
   begin
      if X > 100 then
         F := X - 10      { F ∈ [91, maxint - 10] }
      else
         F := F(F(F(F(X + 33))));     { F ∈ [91, 93] }
      { F ∈ [91, maxint - 10] }
   end;
begin
   readln(X);   Y := F(X);
   { Y ∈ [91, maxint - 10] }
end.
```

shows that the intermediate intervals for the recursive calls cannot be easily guessed on a syntactic basic.

A correct statement about the usefulness of widenings

The power of the widening/narrowing approach to static program analysis by abstract interpretation is more precisely stated as follows:

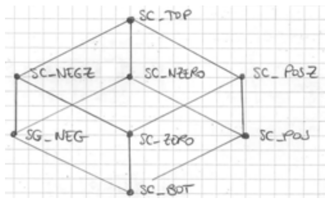
1. For each program there exists a finite lattice which can be used for this program to obtain results equivalent to those obtained using widening/narrowing operators;

2. No such a finite lattice (more precisely, satisfying the ascending chain condition) will do for all programs;
3. For all programs, infinitely many abstract values are necessary;
4. For a particular program it is not possible to infer the set of needed abstract values by a simple inspection of the text of the program.

Another incorrect common believe about the precision of widenings

– It can be thought that an analysis using a more precise abstraction (with widening on an infinite abstract domain not satisfying the ACC) is always more precise than an analysis using a less precise abstraction (e.g. in a finite abstract domain)

– Here is a *counter-example*, using a sign analysis:



– An example of sign analysis is the following:

```
0: { x: SC_TOP}
  x := 1;
2: { x: SC_POS}
  while (x <> 0) do
    4: {x : SC_POS}
      if (0 < x) then
        5: {X: SC_POS}
          x := 0
        6: { x: SC_ZERO}
      else
        7: SC_BOT
          skip
        8: SC_BOT
      fi;
    10: { x: SC_ZERO}
  od
11: { x: SC_ZERO}
```


– The interval analysis with abstract domain

$$\overline{L} \stackrel{\text{def}}{=} \{[\ell,\ u] \mid \ell \in \mathbb{Z} \cup \{-\infty\} \wedge u \in \mathbb{Z} \cup \{+\infty\} \ \wedge\ \ell \leq u\} \cup \{\bot\}$$

and widening extrapolating unstable bounds to infinity:

$$\bot \,\nabla\, X = X$$
$$X \,\nabla\, \bot = X$$
$$[\ell_0,\ u_0] \,\nabla\, [\ell_1,\ u_1] = [(\ell_1 < \ell_0 \ ?\ -\infty : \ell_0),\ (u_1 > u_0 \ ?\ +\infty \ell u_0)]$$

is less precise!

– Program analysis:

```
0: { x:[-∞,+∞]
  x := 1;
2: { x:[-∞, 1]
  while (x <> 0) do
    4: {x:[-∞,1]}
      if (0 < x) then
        5: { x:[1,1] }
          x := 0
        6: { x:[0,0] }
      else
        7:{x:[-∞,0]}
          skip
        8:{x:[-∞,0]}
      fi;
    10: { x:[-∞,0]}
  od
11: { x:[0,0]}
```

```
0 : {x : [-∞,+∞]}
2 : {x : [1,1]}
4 : {x : [1,1]}
5 : {x : [1,1]}
6 : {x : [0,0]}
10 : {x : [0,1]}
2 : {x : [-∞,1]}
4 : {x : [-∞,1]}
5 : {x : [1,1]}
6 : {x : [0,0]}
7 : {x : [-∞,0]}
8 : {x : [-∞,0]}
10 : {x : [-∞,0]}
2 : {x : [-∞,1] ∇ [-∞,1]}
  = {x : [-∞,1]}
stable
```


– The widening is at the origin of the loss of precision:

	previous iterate X	$F(X) = X - 1$	next iterate
signs	SC_POS	SC_POSZ	SC_POSZ
intervals	$[1, +\infty]$	$[0, +\infty]$	$[-\infty, +\infty]$

The 0 threshold is missed by the widening but catched by the sign analysis.

– The interval analysis with improved widening is as precise or more precise than the sign analysis:

```
0: { x:_O_ }
  x := 1;
2: { x:[0,1] }
  while (x <> 0) do
    4: {                  x:[1,1] }        ←    x = 1
      if (0 < x) then
        5: { x:[1,1] }
          x := 0
        6: { x:[0,0] }
      else {(x <= 0)}
        7: _|_
          skip
        8: _|_
      fi;
    10: { x:[0,0] }
  od {(x = 0)}
11: { x:[0,0] }
```


A correct statement about the relative precision of widenings

Theorem. Assume that $\langle L, \sqsubseteq\rangle$, $\langle \overline{L}, \overline{\sqsubseteq}\rangle$ and $\langle \overline{\overline{L}}, \overline{\overline{\sqsubseteq}}\rangle$ are posets such that

(a) $\overline{F} \in \overline{L} \mapsto \overline{L}$, $\overline{\overline{F}} \in \overline{\overline{L}} \mapsto \overline{\overline{L}}$

(b) $\overline{\gamma} \in \overline{L} \mapsto L$, $\overline{\overline{\gamma}} \in \overline{\overline{L}} \mapsto L$

(c) $\overline{a} \in \overline{L}$, $\overline{\overline{a}} \in \overline{\overline{L}}$ with $\overline{\gamma}(\overline{a}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{a}})$

(d) $\overline{\nabla} \in \overline{L} \times \overline{L} \mapsto \overline{L}$, $\overline{\overline{\nabla}} \in \overline{\overline{L}} \times \overline{\overline{L}} \mapsto \overline{\overline{L}}$ satisfy

(d.1) $[\overline{\gamma}(X) \sqsubseteq \overline{\overline{\gamma}}(X') \wedge \overline{\gamma}(Y) \sqsubseteq \overline{\overline{\gamma}}(Y')] \Longrightarrow [\overline{\gamma}(X \mathbin{\overline{\nabla}} Y) \sqsubseteq \overline{\overline{\gamma}}(X' \mathbin{\overline{\overline{\nabla}}} Y')]$

(d.2) $[Y \mathrel{\overline{\sqsubseteq}} X \Longrightarrow X \mathbin{\overline{\nabla}} Y = X]$, $[Y \mathrel{\overline{\overline{\sqsubseteq}}} X \Longrightarrow X \mathbin{\overline{\overline{\nabla}}} Y = X]$

(e) $[\overline{\gamma}(X) \sqsubseteq \overline{\overline{\gamma}}(X')] \Longrightarrow [\overline{\gamma} \circ \overline{F}(X) \sqsubseteq \overline{\overline{\gamma}} \circ \overline{\overline{F}}(X')]$

then the limit $\overline{X}^{\ell}$ of the iterates of $\overline{F}$ from $\overline{a}$ with widening $\overline{\nabla}$ is more precise than the limit $\overline{\overline{X}}^{\ell}$ of the iterates of $\overline{\overline{F}}$ from $\overline{\overline{a}}$ with widening $\overline{\overline{\nabla}}$:

$$\overline{\gamma}(\overline{X}^{\ell}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{\ell})$$

■

Proof. We let $\langle \overline{X}^k, k \geq 0\rangle$ be defined as follows:

$$\begin{cases} \overline{X}^0 = \overline{a} & \\ \overline{X}^{n+1} = \overline{X}^n & \text{if } \overline{F}(\overline{X}^n) \mathrel{\overline{\sqsubseteq}} \overline{X}^n \\ \overline{X}^{n+1} = \overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n) & \text{otherwise} \end{cases}$$

and similarly $\langle \overline{\overline{X}}^k, k \geq 0\rangle$ is defined as follows:

$$\begin{cases} \overline{\overline{X}}^0 = \overline{\overline{a}} & \\ \overline{\overline{X}}^{n+1} = \overline{\overline{X}}^n & \text{if } \overline{\overline{F}}(\overline{\overline{X}}^n) \mathrel{\overline{\overline{\sqsubseteq}}} \overline{\overline{X}}^n \\ \overline{\overline{X}}^{n+1} = \overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n) & \text{otherwise} \end{cases}$$

– We have $\overline{\gamma}(\overline{X}^0) = \overline{\gamma}(\overline{a}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{a}}) = \overline{\overline{\gamma}}(\overline{\overline{X}}^0)$ by (c).

– Assume by induction hypothesis that $\overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$

  - If both iterates have converged then $\overline{\gamma}(\overline{X}^{n+1}) = \overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n) = \overline{\overline{\gamma}}(\overline{\overline{X}}^{n+1})$

  - If $\langle \overline{X}^k, k \geq 0\rangle$ has converged at rank n, but not $\langle \overline{\overline{X}}^k, k \geq 0\rangle$. We have

$\overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$ $\wr$ind. hyp.$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (d.1)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^{n+1}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{n+1})$ $\wr$by def. iterates$\wr$

- If $\langle \overline{\overline{X}}^k,\ k \geq 0\rangle$ has converged at rank $\overline{\ell} \leq n$ but not $\langle \overline{X}^k,\ k \geq 0\rangle$, we have

$\overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$ $\wr$ind. hyp.$\wr$

$\Longrightarrow \overline{\gamma}(\overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (e)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (d.1)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (d.2) since $\langle \overline{\overline{X}}^k,\ k \geq 0\rangle$ has converged at rank $\overline{\ell} \leq n$ and so $\overline{X}^{\overline{\ell}} = \ldots = \overline{X}^n$ and $\overline{F}(\overline{X}^n) \mathrel{\overline{\sqsubseteq}} \overline{X}^n$$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^{n+1}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{n+1})$ $\wr$by def. iterates$\wr$

- If $\langle \overline{X}^k,\ k \geq 0\rangle$ has converged at rank $\overline{\ell} \leq n$ but not $\langle \overline{\overline{X}}^k,\ k \geq 0\rangle$, we have

$\overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$ $\wr$ind. hyp.$\wr$

$\Longrightarrow \overline{\gamma}(\overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (e)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (d.1)$\wr$

$\Longrightarrow$ $\wr$by (d.2) since $\overline{\overline{X}}^{\overline{\overline{\ell}}} = \ldots = \overline{\overline{X}}^n$ with $\overline{\overline{F}}(\overline{\overline{X}}^n) \mathrel{\overline{\overline{\sqsubseteq}}} \overline{\overline{X}}^n$$\wr$
$\overline{\gamma}(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$

$\Longrightarrow \overline{\gamma}(\overline{X}^{n+1}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{n+1})$ $\wr$def. iterates$\wr$

- If none of the $\langle \overline{X}^k,\ k \geq 0\rangle$ and $\langle \overline{\overline{X}}^k,\ k \geq 0\rangle$ have converged at rank n, then:

$\overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$ $\wr$ind. hyp.$\wr$

$\Longrightarrow \overline{\gamma}(\overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (e)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n \mathbin{\overline{\overline{\nabla}}} \overline{\overline{F}}(\overline{\overline{X}}^n))$ $\wr$by (e)$\wr$

$\Longrightarrow \overline{\gamma}(\overline{X}^{n+1}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{n+1})$ $\wr$def. iterates$\wr$

– By recurrence, $\forall n \in \mathbb{N} : \overline{\gamma}(\overline{X}^n) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^n)$

– If $\langle \overline{X}^k,\ k \geq 0\rangle$ converges at rank $\overline{\ell}$ and and similarly $\langle \overline{\overline{X}}^k,\ k \geq 0\rangle$ converges at rank $\overline{\overline{\ell}}$, we have $\overline{\gamma}(\overline{X}^{\overline{\ell}} = \overline{\gamma}(\overline{X}^{\max(\overline{\ell},\overline{\overline{\ell}})}) \sqsubseteq \overline{\overline{\gamma}}(\overline{\overline{X}}^{\max(\overline{\ell},\overline{\overline{\ell}})}) = \overline{\overline{\gamma}}(\overline{\overline{X}}^{\overline{\overline{\ell}}})$.

$\square$

Note: If one iterate has no widening, we can just replace it by the join since F and $\lambda X \cdot X \sqcup F(X)$ have identical iterates when starting from a prefixpoint and F is monotone or F is extensive.

Weakening the hypotheses on widenings (expression of the upper bound overapproximation in term of concretization, no need for lub overapproximation)

– We have shown that for a monotonic $\overline{F}$ on a cpo $\langle \overline{L},\ \overline{\sqsubseteq},\ \overline{\bot},\ \overline{\sqcup}\rangle$, **lfp** $\overline{F}$ is overapproximated by the limit of an upper iteration of $\overline{F}$ from $\overline{\bot}$ with widening $\overline{\nabla}$

– With this point of view, the correctness conditions for the widening are expressed in the abstract

$$X \mathrel{\overline{\sqsubseteq}} X \mathbin{\overline{\nabla}} Y$$
$$Y \mathrel{\overline{\sqsubseteq}} X \mathbin{\overline{\nabla}} Y$$

– In practice, we only want to compare iterations of $F \in L \stackrel{m}{\longmapsto} L$ on the cpo $\langle L, \sqsubseteq, \bot, \sqcup\rangle$ with the iterates of $\overline{F} \in \overline{L} \mapsto \overline{L}$ with widening $\overline{\nabla}$.
– Then the above overapproximation hypotheses can be replaced with

$$\gamma(X) \sqsubseteq \gamma(X \mathbin{\overline{\nabla}} Y)$$
$$\gamma(Y) \sqsubseteq \gamma(X \mathbin{\overline{\nabla}} Y)$$

(together with $F \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ \overline{F}$)

– These hypotheses may be useful when the widening is used both to
- overapproximate non-existent lubs
- accelerate the convergence of the iterates
– The widening $\overline{\nabla}$ is used to generate induction hypotheses which are checked by the convergence condition $\overline{F}(X) \mathrel{\overline{\sqsubseteq}} X$ so no condition on $\overline{\nabla}$ relative to soundness is indeed needed!

Revisiting the soundness of increasing iterations with widening (enforcing convergence without (concrete) lub overapproximation)

THEOREM. [12] Let $F \in L \stackrel{m}{\longmapsto} L$ be a monotone operator on the cpo $\langle L, \sqsubseteq, \sqcap\rangle$. Assume $\bot \in L$ satisfies $\bot \sqsubseteq F(\bot)$. Let $\langle \overline{L}, \overline{\sqsubseteq}\rangle$ be a poset and $\overline{F} \in \overline{L} \mapsto \overline{L}$ such that $F \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ \overline{F}$ where $\gamma \in \overline{L} \stackrel{m}{\longmapsto} L$. Assume that the widening $\overline{\nabla} \in \overline{L} \times \overline{L} \mapsto \overline{L}$ satisfies:

– $\forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \mathbin{\overline{\nabla}} y)$;
– $\forall x, y \in \overline{L} : \gamma(x) \sqsubseteq \gamma(x \mathbin{\overline{\nabla}} y)$ or $\overline{F}$ is extensive, i.e.: $\forall x, y \in \overline{L} : x \mathrel{\overline{\sqsubseteq}} \overline{F}(x)$.

Assume that the widening iteration sequence for $\overline{F}$ from $\overline{\bot}$ (satisfying $\bot \sqsubseteq \gamma(\overline{\bot})$) is $\langle X^n, n \in \mathbb{N}\rangle$, which is defined as follows:

– $\overline{X}^0 = \overline{\bot}$ (a)
– $\overline{X}^{n+1} = \overline{X}^n$ if $\overline{F}(\overline{X}^n) \mathrel{\overline{\sqsubseteq}} \overline{X}^n$ (b)
– $\overline{X}^{n+1} = \overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)$ otherwise (c)

is ultimately stationary at rank $\overline{\ell} \in \mathbb{N}$. Then $\gamma(\overline{F}(\overline{X}^{\ell})) \sqsubseteq \gamma(\overline{X}^{\ell})$ and $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$. ■

PROOF. First observe that $\langle \gamma(\overline{X}^n), n \in \mathbb{N}\rangle$ is an increasing chain since for $\overline{X}^n$ either (b) holds in which case this is trivial by reflexivity since $\gamma(\overline{X}^n) = \gamma(\overline{X}^{n+1})$ or (c) holds, in which case either $\gamma(\overline{X}^n) \sqsubseteq \gamma(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) = \gamma(\overline{X}^{n+1})$ or $\overline{X}^n \mathrel{\overline{\sqsubseteq}} \overline{F}(\overline{X}^n)$ by extensivity and so by monotony $\gamma(\overline{X}^n) \sqsubseteq \gamma(\overline{F}(\overline{X}^n)) \sqsubseteq \gamma(\overline{X}^n \mathbin{\overline{\nabla}} \overline{F}(\overline{X}^n)) = \gamma(\overline{X}^{n+1})$.

[12] Observe that the absence of lub existence hypotheses in $\overline{L}$, that $\overline{F}$ is not assumed to be monotone or extensive and that the widening is only assumed to ensure convergence not to overapproximate lubs.

$\overline{X}^{\overline{\ell}}$ exists by the convergence enforcement hypothesis on the widening. Moreover $\overline{\ell} \geq 1$ since at least one iteration is necessary to check for stability. In case $\overline{X}^{\overline{\ell}}$ satisfies (b), we have

$$\overline{F}(\overline{X}^{\overline{\ell}}) \sqsubseteq \overline{X}^{\overline{\ell}}$$
$$\Longrightarrow \gamma \circ \overline{F}(\overline{X}^{\overline{\ell}}) \sqsubseteq \gamma(\overline{X}^{\overline{\ell}}) \quad \wr \gamma \text{ monotone} \wr$$
$$\Longrightarrow F \circ \gamma(\overline{X}^{\overline{\ell}}) \sqsubseteq \gamma(\overline{X}^{\overline{\ell}}) \quad \wr \text{since } F \circ \gamma \,\dot{\sqsubseteq}\, \gamma \circ \overline{F} \wr$$
$$\Longrightarrow \mathsf{lfp}^{\sqsubseteq}_{\perp} F \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$$

by transfinite induction on the iterates of F from $\perp$ as follows:

- $X^0 = \perp \sqsubseteq \gamma(\overline{\perp}) = \overline{X}^0 \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$ by hypothesis and $\langle \gamma(\overline{X}^n),\ n \in \mathbb{N} \rangle$ is an increasing chain
- If $X^\delta \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$ by induction hypothesis then by monotony of F, we have $X^{\delta+1} = F(X^\delta) \sqsubseteq F \circ \gamma(\overline{X}^{\overline{\ell}}) \sqsubseteq \gamma \circ \overline{F}(\overline{X}^{\overline{\ell}}) \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$ by the convergence condition $\overline{F}(\overline{X}^n) \,\overline{\sqsubseteq}\, \overline{X}^n$ and γ monotone.
- If λ is a limit ordinal and $X^\delta \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$ for all $\delta < \lambda$ by induction hypothesis, then $X^\lambda = \bigsqcup_{\delta<\lambda} X^\delta \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$ by def. of lubs which exist for chains in cpos.
- There exists ϵ such that $\mathsf{lfp}^{\sqsubseteq}_{\perp} F = X^\epsilon \sqsubseteq \gamma(\overline{X}^{\overline{\ell}})$.

Otherwise $\overline{X}^{\overline{\ell}}$ satisfies (c) and we have

$$\overline{X}^{\ell} = \overline{X}^{\ell} \,\overline{\nabla}\, \overline{F}(\overline{X}^{\ell}) \quad \wr \text{by (c) since } \overline{X}^{\overline{\ell+1}} = \overline{X}^{\overline{\ell}} \wr$$
$$\Longrightarrow \gamma(\overline{F}(\overline{X}^{\overline{\ell}})) \sqsubseteq \gamma(\overline{X}^{\ell}) \quad \wr \text{since } \forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \,\overline{\nabla}\, y) \wr$$
$$\Longrightarrow \mathsf{lfp}^{\sqsubseteq}_{\perp} F \sqsubseteq \gamma(\overline{X}^{\overline{\ell}}) \quad \wr \text{as shown above} \wr$$

□

Why widenings cannot be monotone

- Let X and Y be such that $X \sqsubseteq Y$ (e.g. $X \sqsubseteq Y = F(X)$ since the iterates for F with widening ∇ are increasing)
- Assume that ∇ is monotone, we have

$$X \,\nabla\, Y \sqsubseteq Y \,\nabla\, Y$$

- We have seen that is reasonable to assume that $(Y \sqsubseteq X) \Longrightarrow (X \,\nabla\, Y = Y)$ (since e.g. if $Y = F(X) \sqsubseteq X$ then we have converged so their should be no other loss of information)

- In particular for $X = Y$, we have

$$Y \,\nabla\, Y = Y$$

- It follows that

$$X \,\nabla\, Y \sqsubseteq Y$$

which prevents extrapolations!

Example of non-monotone widening

– the classical widening on intervals is:

$$\bot \nabla X = X \nabla \bot = X$$
$$[\ell_0,\ u_0] \nabla [\ell_1,\ u_1] = [(\ell_1 < \ell_0\ ?\ -\infty \ ¿\ \ell_0),\ (u_1 > u_0\ ?\ +\infty\ ¿\ u_0)]$$

– Not monotone in its first argument: $[0,1] \sqsubseteq [0,2]$ but $[0,1] \nabla [0,2] = [0,+\infty] \not\sqsubseteq [0,2] = [0,2] \nabla [0,2]$

– Monotone in its second parameter: $(I' \sqsubseteq I'') \Longrightarrow (I \nabla I' \sqsubseteq I \nabla I'')$

Proof. - If $I = \bot$: $(I \nabla I' = \bot \nabla I' = I' \sqsubseteq I'' = \bot \nabla I'' = I \nabla I''$

- Else $I = [a,b] \neq \bot$. Then:
  · If $I' = \bot$ then $I \nabla I' = I \nabla \bot = I \sqsubseteq I \nabla I''$
  · Else $I' = [a',b'] \neq \bot$ so $I' \sqsubseteq I''$ implies $I'' = [a'',b''] \neq \bot$ with $a'' \leq a'$ and $b' \leq b''$.
    - For the lower bound, we have:
      - If $a' < a$ so $a'' \leq a' < a$ hence we have $I \nabla I' = [a,\_] \nabla [a',\_] = [-\infty,\_] \sqsubseteq I \nabla I'' = [a,\_] \nabla [a'',\_] = [-\infty,\_]$
      - Else, $a'' \geq a$, hence $I \nabla I' = [a,\_] \nabla [a',\_] = [a,\_] \sqsubseteq I \nabla I'' = [a,\_] \nabla [a'',\_] = (a'' \geq a\ ?\ [a,\_]\ ¿\ [-\infty,\_])$
    - Idem, for the upper bound.

□

Consequences of the absence of monotony of the widening

A *local improvement* of a static analysis may lead to a *global deterioration* of the precision.

– Example:

```
    X := 0;
{1}
    while true do
{2}
       if X = 0 then
{3}
         X := 1
{4}
       else
{5}
          X := 2
{6}
       fi
{7}
    od
```


– Analysis 1: $X = 0 \Longrightarrow X \in [0,2]$, locally imprecise

  1 $\rightarrow X \in [0,2]$ because of local imprecision
  2 $\rightarrow X \in \bot \nabla [0,2] = [0,2]$
  3 $\rightarrow X \in [0,0]$
  4 $\rightarrow X \in [1,1]$
  5 $\rightarrow X \in [1,2]$
  6 $\rightarrow X \in [2,2]$
  7 $\rightarrow X \in [1,1] \sqcup [2,2] = [1,2]$
  2 $\rightarrow X \in [0,2] \nabla [1,2] = [0,2]$, stable!

- Analysis 2: $X = 0 \Longrightarrow X \in [0,0]$, locally precise
  - $1 \rightarrow X \in [0,0]$ because of local precision
  - $3 \rightarrow X \in [0,0]$
  - $4 \rightarrow X \in [1,1]$
  - $5 \rightarrow \bot$
  - $6 \rightarrow \bot$
  - $7 \rightarrow X \in \bot \sqcup [1,1] = [1,1]$
  - $2 \rightarrow X \in [0,0] \nabla ([0,0] \sqcup [1,1]) = [0,+\infty]$
  - $3 \rightarrow X \in [0,0]$
  - $4 \rightarrow X \in [1,1]$
  - $5 \rightarrow [1,+\infty]$
  - $6 \rightarrow [2,2]$

  - $7 \rightarrow X \in [1,1] \sqcup [2,2] = [1,2]$
  - $2 \rightarrow X \in [0,+\infty] \nabla ([0,0] \sqcup [1,2]) = [0,+\infty]$, stable
- Remedies:
  - Use a *narrowing*, if possible
    - In the example, $[0,+\infty] \Delta [0,2] = [0,2]$ so that the final result is exact
    - The narrowing is not always able to compensate for the lack of precision of the widening, so a stable bound can be missed
  - Choose a *more precise widening*

Revisiting the soundness of decreasing iterations with narrowing

THEOREM. [13] Let $F \in L \stackrel{m}{\longmapsto} L$ be a monotone operator on the cpo $\langle L, \sqsubseteq, \sqcap\rangle$. Let $\langle \overline{L}, \overline{\sqsubseteq}\rangle$ be a poset. Let $\gamma \in \overline{L} \stackrel{m}{\longmapsto} L$ be a monotone concretization function. Let $\overline{F} \in \overline{L} \mapsto \overline{L}$ such that $F \circ \gamma \dot{\sqsubseteq} \gamma \circ \overline{F}$. Let $A \in \overline{L}$ be such that $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(A)$. Assume that the narrowing $\Delta \in \overline{L} \times \overline{L} \mapsto \overline{L}$ satisfies:

- $\forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \,\Delta\, y)$.

(which can be restricted to the case $y \,\overline{\sqsubseteq}\, x$ and even $y = \overline{F}(x)$)
Assume that the narrowing iteration sequence for $\overline{F}$ from A defined as

- $\overline{Y}^0 = \overline{F}(A)$ [14] (a)

- $\overline{Y}^{n+1} = \overline{Y}^n$ if $\overline{Y}^n \,\overline{\sqsubseteq}\, \overline{F}(\overline{Y}^n)$ (b)
- $\overline{Y}^{n+1} = \overline{Y}^n \,\overline{\Delta}\, \overline{F}(\overline{Y}^n)$ otherwise (c)

is ultimately stationary at rank $\ell \in \mathbb{N}$. Then $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(\overline{Y}^{\ell})$. ■

PROOF. We prove that $\forall n \in \mathbb{N} : \mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(\overline{Y}^n)$ so that the narrowing iteration can be stopped at any iteration rank [15].

- For the basis, we have $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(A)$ by hypothesis (which results from the widening phase) and, so by fixpoint property, monotony of F, $F \circ \gamma \dot{\sqsubseteq} \gamma \circ \overline{F}$ and (a), we have $\mathsf{lfp}_{\bot}^{\sqsubseteq} F = F(\mathsf{lfp}_{\bot}^{\sqsubseteq} F) \sqsubseteq F(\gamma(A)) \sqsubseteq \gamma \circ \overline{F}(A) = \gamma(\overline{Y}^0)$

[13] Observe that $\overline{F}$ is not assumed to be monotone. If it is extensive, a narrowing is of no interest.
[14] $\overline{F}(A)$ has already been computed to stop the widening iteration so that it would be less efficient to restart from A.
[15] so e.g. the narrowing can be $x \,\Delta\, y = y$ and the iteration restricted to one step.

– Assume that $n \leq \ell$ and, by induction hypothesis, $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(\overline{Y}^n)$. There are two cases according to the definition of the iterates:
  - If $\overline{Y}^n \mathrel{\overline{\sqsubseteq}} \overline{F}(\overline{Y}^n)$ then $\overline{Y}^{n+1} = \overline{Y}^n$ so that by induction hypothesis $\mathsf{lfp}_{\bot}^{\sqsubseteq} F \sqsubseteq \gamma(\overline{Y}^{n+1})$ and indeed, $n = \ell$.

– Otherwise, $n < \ell$ and $\overline{Y}^n \mathrel{\Delta} \overline{F}(\overline{Y}^n)$. In that case, we have

$$
\begin{array}{ll}
\mathsf{lfp}_{\bot}^{\sqsubseteq} F & \\
= F(\mathsf{lfp}_{\bot}^{\sqsubseteq} F) & \wr \text{fixpoint property} \wr \\
\sqsubseteq F(\gamma(\overline{Y}^n)) & \wr \text{ind. hyp. and } F \text{ monotone} \wr \\
\sqsubseteq \gamma(\overline{F}(\overline{Y}^n)) & \wr \text{by } F \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ \overline{F} \text{ and transitivity} \wr \\
\sqsubseteq \gamma(\overline{Y}^n \mathrel{\Delta} \overline{F}(\overline{Y}^n)) & \wr \text{by } \forall x, y \in \overline{L} : \gamma(y) \sqsubseteq \gamma(x \mathrel{\Delta} y) \wr \\
= \gamma(\overline{Y}^{n+1}) & \wr \text{def. iterates} \wr
\end{array}
$$

We conclude by recurrence on n, noting that the iterates are stationary beyond ℓ. □

Design of Widening/Narrowing

Strategies to improve the precision of iterations with widening/narrowing (iteration threshold, unrolling, cut-points, history-based extrapolation)

– *Iteration threshold*: do not widen/narrow in the first iterations (e.g. in a loop), up to some threshold n

– *Unrolling*: semantically unroll the first iterates of a loop, so that, e.g.:

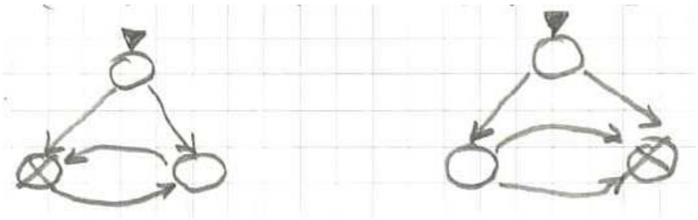
```
B := true;
while true do if B then I else C; B := false od
```

as found in some automatically generated code will be handled as:

```
I; while true do C od
```


Done n times, this is more precise than a temporization with iteration threshold since no join is performed at all in the first iterations

– Widening/narrowing/stabilization checks at *cut-points* only
  - Minimal number of cut-points
  - A cut point within each loop (more precisely, within each circular dependency)
  - The choice may not be unique (for irreducible dependence graphs)



– *Computation history-based extrapolation*:

A simple example:

- Do not widen/narrow if a component of the system of fixpoint equations was computed for the first time since the last widening/narrowing ;
- Otherwise, do not widen/narrow the abstract values of variables which were not "assigned to" [16] since the last widening/narrowing.

– Example:

[16] more precisely which did not appear in abstract equations corresponding to an assignment to these variables.

- With widening/narrowing at cut-points:

```
{ i:_O_; j:_O_ }
  i := 1;
{ i:[1,+oo]; j:[1,+oo]? }
  while (i < 1000) do
    { i:[1,999]; j:[1,+oo]? }
      j := 1;
    { i:[1,+oo]; j:[1,+oo] }
      while (j < i) do
        { i:[2,+oo]; j:[1,1073741822] }
          j := (j + 1)
        { i:[2,+oo]; j:[2,+oo] }
      od;
    { i:[1,+oo]; j:[1,+oo] }
      i := (i + 1);
    { i:[2,+oo]; j:[1,+oo] }
  od
{ i:[1000,+oo]; j:[1,+oo]? }
```


- With history-based widening/narrowing:

```
{ i:_O_; j:_O_ }
  i := 1;
{ i:[1,1000]; j:[1,999]? }
  while (i < 1000) do
    { i:[1,999]; j:[1,999]? }
      j := 1;
    { i:[1,999]; j:[1,999] }
      while (j < i) do
        { i:[2,999]; j:[1,998] }
          j := (j + 1)
        { i:[2,999]; j:[2,999] }
      od;
    { i:[1,999]; j:[1,999] }
      i := (i + 1);
    { i:[2,1000]; j:[1,999] }
  od
{ i:[1000,1000]; j:[1,999]? }
```

– More generally, the extrapolation is more precise if we:
  - widen up to constants, ranges, ... given by declarations, tests, ...;
  - have the widening depend upon the iteration step, e.g. by:
    · introducing a threshold under which the least upper bound is used and above which widening is enforced;
    · awaiting for regular behaviors before widening within loops:
      - do not widen on the first iterate,
      - do not widen if a new branch of a test has just been taken within the loop body.

Thresholded/layered widening

Let $\langle L, \sqsubseteq, \bot, \top, \sqcup, \sqcap\rangle$ be a complete lattice. Let $T_1 \sqsubset T_2 \sqsubset \ldots : \sqsubset T_n = \top$ be finitely many elements of L. Define $T = \{T_1, \ldots, T_n\}$. The widening with thresholds T is

$$X \mathbin{\nabla_T} Y = (\!| Y \sqsubseteq X \mathrel{?} X \mathrel{¿} T_i |\!)$$
$$\text{where} \quad X \sqcup Y \sqsubseteq T_i$$
$$\text{and} \quad \forall T_j \in T : X \sqcup Y \sqsubseteq T_j \Longrightarrow T_i \sqsubseteq T_j$$

THEOREM. $X \mathbin{\nabla_T} Y$ is a widening. ■

PROOF. 1. ∇_T is a upper bound.
– If $Y \sqsubseteq X$ then $X \mathbin{\nabla_T} Y = X = X \sqcup Y$;
– Otherwise, $Y \sqsubseteq X = T_i \sqsupseteq X \sqcup Y$.

2. ∇_T enforces convergence. Given $\langle X_i,\ i \in \mathbb{N}\rangle$ define $Y_0 = X_0, \ldots, Y_{i+1} = Y_i \mathbin{\nabla} X_i$. Assume that $\langle Y_i,\ i \in \mathbb{N}\rangle$ is a strictly increasing chain. We have $Y_1 = X_0 \mathbin{\nabla_T} X_1 = T_{i_1}$ (since otherwise $X_1 \sqsubseteq X_0$ and $Y_1 = X_0$ so $\langle Y_i,\ i \in \mathbb{N}\rangle$ would not be a strictly increasing chain). Assume by induction hypothesis that $Y_k = T_{i_k}$ with $T_{i_1} \sqsubset \ldots \sqsubset T_{i_k}$. Then $Y_{k+1} = T_{i_k} \mathbin{\nabla_T} X_{k+1}$ since we cannot have $X_{k+1} \sqsubseteq Y_{k+1}$ which would imply that $Y_{k+1} = T_{i_k}$ in contradiction with the hypothesis that $\langle Y_i,\ i \in \mathbb{N}\rangle$ is a strictly increasing chain. So $Y_{k+1} = T_{i_{k+1}}$ with $T_{i_{k+1}} \sqsupseteq T_{i_k}$. But $T_{i_{k+1}} \neq T_{i_k}$ since otherwise $\langle Y_i,\ i \in \mathbb{N}\rangle$ would not be a strictly increasing chain. It follows, by recurrence that $\forall k \in \mathbb{N} : Y_k = T_{i_k}$ so $\langle T_{i_k},\ k \in \mathbb{N}\rangle$ is strictly increasing, a contradiction.

□

Widenings for pairs/tuples

– If
  - ∇_1 is a widening for $\langle L_1, \sqsubseteq_1\rangle$, and
  - ∇_2 is a widening for $\langle L_2, \sqsubseteq_2\rangle$,

  then

$$\langle x,\ y\rangle \mathbin{\nabla} \langle x',\ y'\rangle \stackrel{\text{def}}{=} \langle x \mathbin{\nabla_1} x',\ y \mathbin{\nabla_2} y'\rangle$$

  is a widening for $\langle L_1 \times L_2, \sqsubseteq_1 \times \sqsubseteq_2\rangle$ where $\sqsubseteq_1 \times \sqsubseteq_2$ is the componentwise ordering
– Idem for narrowing
– Idem for tuples

First-order functional widening

As we have seen, if:

- $f \in L \stackrel{\mathrm{m}}{\longmapsto} L$, ∇ is a widening on a poset $\langle L, \sqsubseteq\rangle$

then

$$\mathsf{lfp}^{\sqsubseteq} f \sqsubseteq \mathsf{lfp}^{\sqsubseteq} \lambda x \cdot x \mathbin{\nabla} f(x)$$

(and the second fixpoint can be computed iteratively starting from a prefixpoint $\bot \sqsubseteq f(\bot)$ in finitely many steps).

Example: Interval Analysis of Functions

Solve the second-order equation:

$$f = F(f) \text{ where } f(x) = [1,1] \sqcup (f(x) + [2,2])$$

for the argument $[0,0]$.

So we approximate by:

$$f = f \mathbin{\nabla} F(f) \text{ for argument } [0,0],$$

that is:

$$f([0,0]) = f([0,0]) \mathbin{\nabla_1} F(f([0,0]))$$
$$= f([0,0]) \mathbin{\nabla_1} F([1,1] \sqcup (f([0,0]) + [2,2]))$$

We let $X \stackrel{\mathrm{def}}{=} f([0,0])$ so we get a first order equation:

$$X = X \mathbin{\nabla_1} ([1,1] \sqcup (X + [2,2]))$$

The equation is solved iteratively as follows:

$X^0 = \bot$
$X^1 = X^0 \mathbin{\nabla_1} ([1,1] \sqcup (X^0 + [2,2])) = [1,1]$
$X^2 = X^1 \mathbin{\nabla_1} ([1,1] \sqcup (X^1 + [2,2]))$
$= [1,1] \mathbin{\nabla_1} ([1,1] \sqcup ([1,1] + [2,2]))$
$= [1,1] \mathbin{\nabla_1} [1,3] = [1,+\infty]$
$X^2 = [1,1] \mathbin{\nabla_1} [1,+\infty] = [1,+\infty]$
proving that $f(0)$ is greater than 1.

Second-order functional widening – I – finite domains

If

- $F \in (S \mapsto L) \stackrel{\mathrm{m}}{\longmapsto} (S \mapsto L)$, pointwise ordering;
- ∇ is a widening for $\langle L, \sqsubseteq\rangle$;
- S is a finite set

then

$$\mathsf{lfp}^{\dot\sqsubseteq} F \mathrel{\dot\sqsubseteq} \mathsf{lfp}^{\dot\sqsubseteq} \lambda f \cdot \lambda x \cdot f(x) \mathbin{\nabla} F(f)(x)$$

Note:
This can be seen as a system of equations:

$$\begin{cases} X_i = F_i(X_1, \ldots, X_n) \\ \quad i = 1, \ldots, n \end{cases}$$

where $S = \{1, \ldots, n\}$ and X_i is $f(i)$.
This is solved as:

$$\begin{cases} X_i = X \mathbin{\nabla} F_i(X_1, \ldots, X_n) \\ \quad i = 1, \ldots, n \end{cases}$$

with the usual remark that ∇ is needed only once around cycles of the dependence graph.

Possible divergence for infinite domains

– Note: the previous widening strategy fails for

$$f(x) = [1,1] \sqcup (f(x + [1,1]) + [2,2])$$

since $f([0,0])$ needs $f([1,1])$ which needs $f([2,2])$, etc.

Second-order functional widening – II – infinite domains

– If
- ∇_1 is a widening on $\langle L_1, \sqsubseteq_1 \rangle$,
- ∇_2 is a widening on $\langle L_2, \sqsubseteq_2 \rangle$
- $F \in (L_1 \stackrel{\mathrm{m}}{\longmapsto} L_2) \stackrel{\mathrm{m}}{\longmapsto} (L_1 \stackrel{\mathrm{m}}{\longmapsto} L_2)$

then

$$\mathsf{lfp}^{\dot{\sqsubseteq}_2} F \mathrel{\dot{\sqsubseteq}_2} \mathsf{lfp}\, \lambda f \cdot \lambda x \cdot f(x) \mathbin{\nabla_2} F(\lambda y \cdot f(x \mathbin{\nabla_1} y))(x)$$

where $\dot{\sqsubseteq}_2$ is the pointwise ordering on $L_1 \stackrel{\mathrm{m}}{\longmapsto} L_2$.

Reference

[3] P. Cousot and R. Cousot. Static determination of dynamic properties of recursive procedures. In *IFIP Conf. on Formal Description of Programming Concepts, St-Andrews, N.B., CA*, E.J. Neuhold (Ed.), pages 237–277, St-Andrews, N.B., Canada, 1977. North-Holland Publishing Company (1978).

Example of second-order functional widenings in infinite domains

$$F = \lambda f \cdot \lambda x \in 1, 1 \cdot \sqcup (f(x + [1,1]) + [2,2])$$

$\mathsf{lfp}^{\dot{\sqsubseteq}}_{\lambda x \cdot \bot} F$ is approximated as the least solution to:

$$\begin{aligned} f(x) &= f(x) \mathbin{\nabla_2} F(\lambda y \cdot f(x \mathbin{\nabla_1} y))(x) \\ &= f(x) \mathbin{\nabla_2} ([1,1] \sqcup (\lambda y \cdot f(x \mathbin{\nabla_1} y) x + [1,1]) + [2,2])) \\ &= f(x) \mathbin{\nabla_2} ([1,1] \sqcup (f(x \mathbin{\nabla_1} (x + [1,1])) + [2,2])) \end{aligned}$$

In order to compute $f([0,0])$ we follow a chatotic iteration strategy (see [3]):

– A table of pairs $\langle a,\ f(a)\rangle$ is maintained for needing arguments only, starting from $\langle [0,0],\ \bot\rangle$;

– We recompute $f^{n+1}(a)$ for the pair $\langle a,\ f^n(a)\rangle$ using $f^n(a)$ as the current approximation to $f(a)$

as long as:

– no new argument a' is needed;

– all needed pairs $\langle a,\ f(a)\rangle$ are stable.

- $f^0([0,0]) = \bot$
- $f^1([0,0])$
$= f^0([0,0])\,\nabla_2([1,1] \sqcup (f^0([0,0]\,\nabla_1([0,0]+[1,1]))+[2,2]))$
$= \bot\,\nabla_2\,([1,1] \sqcup (f^0([0,0]\ \nabla_1\ ([0,0]+[1,1]))+[2,2]))$
$= ([1,1] \sqcup (f^0([0,0]\ \nabla_1\ [1,1]) + [2,2]))$
$= ([1,1] \sqcup (f^0([0,+\infty]) + [2,2]))$
$= ([1,1] \sqcup \bot)$
$= [1,1]$
since $f([0,+\infty])$ has not yet been computed, hence:
- $f^0([0,+\infty]) = \bot$ ...

- $f^1([0,+\infty])$
$= f^0([0,+\infty])\nabla_2([1,1] \sqcup (f^0([0,+\infty]\nabla_1([0,+\infty]+[1,1]))+[2,2]))$
$= \bot\,\nabla_2\,([1,1] \sqcup (f^0([0,+\infty]\ \nabla_1\ [1,+\infty]) + [2,2]))$
$= ([1,1] \sqcup (f^0([0,+\infty]\ \nabla_1\ [1,+\infty]) + [2,2]))$
$= ([1,1] \sqcup (f^0([0,+\infty]) + [2,2]))$
$= ([1,1] \sqcup \bot)$
$= [1,1]$ ...

- $f^2([0,+\infty])$
$= f^1([0,+\infty])\nabla_2([1,1] \sqcup (f^1([0,+\infty]\nabla_1([0,+\infty]+[1,1]))+[2,2]))$
$= [1,1]\ \nabla_2\,([1,1] \sqcup (f^1([0,+\infty]) + [2,2]))$
$= [1,1]\ \nabla_2\,([1,1] \sqcup ([1,1] + [2,2]))$
$= [1,1]\ \nabla_2\,([1,1] \sqcup ([3,3]))$
$= [1,1]\ \nabla_2\,[1,3])$
$= [1,+\infty])$ ...

- $f^3([0,+\infty])$
$= f^2([0,+\infty])\nabla_2([1,1]\sqcup(f^2([0,+\infty]\nabla_1([0,+\infty]+[1,1]))+[2,2]))$
$= [1,\infty]\ \nabla_2\ ([1,1]\sqcup(f^2([0,+\infty]) + [2,2]))$
$= [1,\infty]\ \nabla_2\ ([1,1]\sqcup([1,+\infty] + [2,2]))$
$= [1,\infty]\ \nabla_2\ ([1,1]\sqcup([3,\infty]))$
$= [1,\infty]\ \nabla_2\ [1,\infty])$
$= [1,+\infty])$ ...

- $f^2([0,0])$
$= f^1([0,0])\nabla_2([1,1]\sqcup(f^1([0,0]\nabla_1([0,0]+[1,1]))+[2,2]))$
$= [1,1]\ \nabla_2\ ([1,1]\sqcup(f^0([0,\infty]) + [2,2]))$
$= [1,1]\ \nabla_2\ [1,+\infty])$
$= [1,+\infty]$

Everything needed is stable.

Note: This chaotic iteration strategy from [3] was used can be chosen as a semantics of procedures (in the finite case so no widening is needed) by Jones & Mycroft [4] under the popular name "minimal function graphs".

Reference

[4] N. Jones and A. Mycroft. Data flow analysis of applicative programs using minimal function graphs. Annual Symposium on Principles of Programming Languages archive Proceedings of the 13th ACM SIGACT-SIGPLAN symposium on Principles of programming languages table of contents St. Petersburg Beach, Florida, pp. 296–306, 1986.

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- Extension to recursive procedures:

[8] P. Cousot and R. Cousot. Static determination of dynamic properties of recursive procedures. In *IFIP Conf. on Formal Description of Programming Concepts, St-Andrews, N.B., CA*, E.J. Neuhold (Ed.), pages 237–277, St-Andrews, N.B., Canada, 1977. North-Holland Publishing Company (1978).

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[9] P. Cousot. «*Méthodes itératives de construction et d'approximation de points fixes d'opérateurs monotones sur un treillis, analyse sémantique de programmes*». Thèse d'État ès sciences mathématiques, Université scientifique et médicale de Grenoble, Grenoble, 21 mars 1978.

otherwise, see

[10] P. Cousot. Semantic foundations of program analysis. In S.S. Muchnick and N.D. Jones, editors, *Program Flow Analysis: Theory and Applications*, chapter 10, pages 303–342. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, USA, 1981.

– Showing that widening/narrowing is strictly more powerful than abstraction/concretization (and therefore abstract model-checking which is a particular case):

[11] P. Cousot and R. Cousot. Comparing the Galois Connection and Widening/Narrowing Approaches to Abstract Interpretation, invited paper. In M. Bruynooghe and M. Wirsing, eds., *Programming Language Implementation and Logic Programming, Proc. 4th In. Symp., PLILP'92*, Leuven, Belgium, 13–17 Aug. 1992, LNCS 631, p. 269–295. Springer-Verlag, 1992.

– The first use of a concretization-only framework (which is used in this course in absence of abstract lubs whence of best approximations):

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– Making clear the possible choice of abstraction-concretization, abstraction-only and concretization-only frameworks:

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THE END

My MIT web site is http://www.mit.edu/~cousot/

The course web site is http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/.