A Linear Programming Example

Minimum Fuel Spacecraft Rendezvous
(Adapted from notes by Sommer Gentry and Eric Feron)

The optimal rendezvous problem for a spacecraft is to sucessfully dock a spacecraft from a set of initial conditions (position, speed) on a target which is assumed to be immobile. The goal of this note is to show you that such a problem may be converted into an LP, and thus be solved as an optimization problem.

1 Mathematical problem formulation

We assume that the spacecraft can move only in one dimension, and that thrust is available in both forward and backwards directions. Let the avail- able thrust be w and the spacecraft mass be M. Let x be the position of the spacecraft. The equations of motion read

$$M\ddot{x} = w, \tag{1}$$

or, writing $u = w/M$, the EOMs equivalently read

$$\ddot{x} = u \tag{2}$$

with the initial conditions $x(0)$ and $\dot{x}(0)$. In order to correctly formulate the rendezvous problem, we need to know at what time the rendezvous is going to occur, say T. Then we know we must have

$$x(T) = 0, \dot{x}(T) = 0. \tag{3}$$

The objective function choice is obvious: fuel is a very precious good. Assuming that the thrust intensity is proportional to the fuel flow, the total amount of fuel spent is proportional to

$$Z = \int_0^T |u| dt. \tag{4}$$

The goal of the problem is thus to minimize Z. Of course, several constraints need to be added to account for the physical nature of the spacecraft. For example, the fuel flow, and therefore the available thrust or u is limited in intensity at every instant; this constraint is expressed as

$$|u| \leq u_{max}. \tag{5}$$

Also, certain position and speed profiles might not be acceptable. For example, speed may need to be small near rendezvous. In general, there may be a maximum speed profile

$$|\dot{x}(t)| \leq \dot{x}_{max}(t), \tag{6}$$

where $\dot{x}_{max}(t)$ is a predefined bounding function. Likewise, position is constrained to be positive (a negative position corresponds to a collision in this example), or in general, may have to be greater than some specified profile

$$x(t) \geq x_{min}(t). \tag{7}$$

The problem above may be seen as an optimization problem. While u clearly appears as the optimization variable and x, $\dot{x}$ is logically obtained from u using the EOMs (1), the operations research framework would rather see u and x, $\dot{x}$ equally as optimization variables, subject to the linear equality constraint (1).

2 Discretization and conversion to LP

The optimization problem as posed cannot be solved on a computer, because x varies continuously over T. We only know how to choose the continuous value for a specific x_t, not how to choose the values over a continuous (and therefore infinite in number) range of x_t. The first step is therefore to appropriately discretize the problem using a discretization step ΔT. Let N be an integer such that $N\Delta T = T$. Denoting the discretized variables as

$$\begin{aligned} x &= [x_0 = x(0), \ldots, x_N = x(T)] && (8)\\ \dot{x} &= [\dot{x}_0 = \dot{x}(0), \ldots, \dot{x}_N = \dot{x}(T)] && (9)\\ u &= [u_0 = u(0), \ldots, u_{N-1} = u(T - \Delta T)]. && (10) \end{aligned}$$

Constraints due to the Equations of Motion

An approximately discretized version of the EOMs (1) is then

$$x_{k+1} = x_k + \Delta T \dot{x}_k, k = 1, \ldots, N-1, \tag{11}$$

and

$$\dot{x}_{k+1} = \dot{x}_k + \Delta T u_k, k = 1, \ldots, N-1. \tag{12}$$

This may also be seen as a set of equality constraints between x, $\dot{x}$ and u. Considering the large decision vector

$$\mathbf{y} = \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix}, \tag{13}$$

the set of constraints (11) may also be written in matrix form as

$$A_{xEOM}\mathbf{y} = \mathbf{b}_{xEOM} \tag{14}$$

where

$$A_{xEOM} = \left[\begin{array}{ccccc|cccc|ccc} -1 & 1 & 0 & \cdots & 0 & -\Delta T & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & \ddots & 0 & 0 & \vdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \ddots & -1 & 1 & 0 & \cdots & 0 & -\Delta T & 0 & \cdots & 0 \end{array}\right] \tag{15}$$

and $\mathbf{b}_{xEOM} = 0$. Likewise, the set of constraints (12) may also be written in matrix form as

$$A_{\dot{x}EOM}\mathbf{y} = \mathbf{b}_{\dot{x}EOM} \tag{16}$$

where

$$A_{\dot{x}EOM} = \left[\begin{array}{ccc|ccccc|cccc} 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 & -\Delta T & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & 0 & -1 & 1 & \cdots & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \ddots & -1 & 1 & 0 & \cdots & 0 & -\Delta T \end{array}\right] \tag{17}$$

and $\mathbf{b}_{\dot{x}EOM} = 0$.

Constraints due to the Termination Conditions

The initial and final conditions are similarly written as the set of equality constraints

$$A_{term}\mathbf{y} = \mathbf{b}_{term} \tag{18}$$

where

$$A_{term} = \left[\begin{array}{cccc|cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \tag{19}$$

and

$$\mathbf{b}_{term} = \begin{bmatrix} x(0) \\ \dot{x}(0) \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

Constraints due to Power Limitations

The power constraint given in (5) can be formulated as

$$\left.\begin{array}{r} u_k \leq u_{max} \\ -u_k \leq u_{max} \end{array}\right\} \forall u_k \tag{21}$$

and we can write this as the constraint

$$A_{power}\mathbf{y} \leq \mathbf{b}_{power} \tag{22}$$

where

$$A_{power} = \left[\begin{array}{c|c|c} 0 & 0 & I \\ \hline 0 & 0 & -I \end{array}\right] \tag{23}$$

and

$$\mathbf{b}_{power} = u_{max} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{24}$$

Constraints due to Speed Limitations

Similarly, the velocity constraint given in (6) can be formulated as

$$A_{speed}\mathbf{y} \leq \mathbf{b}_{speed} \tag{25}$$

where

$$A_{speed} = \left[\begin{array}{c|c|c} 0 & I & 0 \\ \hline 0 & -I & 0 \end{array}\right] \tag{26}$$

and

$$\mathbf{b}_{speed} = \begin{bmatrix} \dot{x}_{max}(0) \\ \vdots \\ \dot{x}_{max}(N\Delta T) \\ \dot{x}_{max}(0) \\ \vdots \\ \dot{x}_{max}(N\Delta T) \end{bmatrix}. \tag{27}$$

Constraints on Position

Finally, the position constraint in (7) can be represented as

$$A_{position}\mathbf{y} \leq \mathbf{b}_{position} \tag{28}$$

where

$$A_{position} = \left[-I \mid 0 \mid 0 \right] \tag{29}$$

and

$$\mathbf{b}_{position} = \begin{bmatrix} -x_{min}(0) \\ \vdots \\ -x_{min}(N\Delta T) \end{bmatrix}. \tag{30}$$

The Objective Function

The objective function may be easily discretized and represented as

$$Z = \Delta T \sum_{i=0} N - 1|u_i|. \tag{31}$$

The absolute value means that this objective is *not* linear, but it may be transformed into a linear objective with additional linear constraints and additional variables. We introduce the new decision variables $\alpha_0, \ldots, \alpha_{N-1}$ and the additional constraints

$$|u_i| \leq \alpha_i, i = 0, \ldots, n-1. \tag{32}$$

Then minimizing Z is equivalent to minimizing $\sum_{i=1}^{N-1} \alpha_i$ subject to (32). Of course, the constraints (32)are equivalent to the linear constraints

$$u_i - \alpha_i \leq 0 \tag{33}$$
$$-u_i - \alpha_i \leq 0 \tag{34}$$

Thus, the overall linear programming problem is to minimize $\sum_{i=1}^{N-1} \alpha_i$ subject to the constraints (34).