# Mechanics

## Physics 151

Lecture 3

Lagrange’s Equations

(Goldstein Chapter 1)

Hamilton’s Principle

(Chapter 2)

# What We Did Last Time

- Discussed multi-particle systems
  - Internal and external forces
    - Laws of action and reaction
- Introduced constraints
  - Generalized coordinates
- Introduced Lagrange's Equations
  - ... and didn't do the derivation

→ Let's pick it up and start from there

# Today’s Goals

- Derive Lagrange’s Eqn from Newton’s Eqn
  - Use D’Alembert’s principle
  - There will be a few assumptions
    - Will make them clear as we go
- Introduce Hamilton’s Principle
  - Equivalent to Lagrange’s Equations
    - Which in turn is equivalent to Newton’s Equations
  - Does not depend on coordinates by construction
  - Derivation in the next lecture

# Lagrange's Equations

Recipe

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$L(q, \dot{q}, t) \equiv T - V$$

Kinetic energy

Potential energy

*Lagrangian*

- Express $L = T - V$ in terms of generalized coordinates $\{q_j\}$, their time-derivatives $\{\dot{q}_j\}$, and time $t$
  - The potential $V = V(q, t)$ must exist
  - i.e. all forces must be conservative

# Virtual Displacement

- Consider a system with constraints
  - Ordinary coordinates $\mathbf{r}_i$ ($i$ = 1...$N$)
  - Generalized coordinates $q_j$ ($j$ = 1...$n$)

$$\begin{cases} \mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \ldots, q_n, t) \\ \mathbf{r}_2 = \mathbf{r}_2(q_1, q_2, \ldots, q_n, t) \\ \vdots \\ \mathbf{r}_N = \mathbf{r}_N(q_1, q_2, \ldots, q_n, t) \end{cases}$$

- Imagine moving all the particles slightly $\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i$ $\quad q_j \to q_j + \delta q_j$

Virtual displacement

  - Note that $\delta\mathbf{r}_i$ must satisfy the constraints

$$\delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

3$N$ coordinates not independent

$n$ coordinates independent

# D'Alembert's Principle

- From Newton's Equation of Motion

  $\mathbf{F}_i = \dot{\mathbf{p}}_i \Rightarrow \mathbf{F}_i - \dot{\mathbf{p}}_i = 0$

- Part of the force $\mathbf{F}_i$ must be due to constraints

  $$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$$

  "applied" force: $\mathbf{F}_i^{(a)}$; "constraint" force: $\mathbf{f}_i$

  - Applied force is "known" $\mathbf{F}_i^{(a)} = \mathbf{F}_i^{(a)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N, t)$

- Constraint force $\mathbf{f}_i$ (usually) does no work
  - Movement is perpendicular to the force $\mathbf{f}_i \delta\mathbf{r}_i = 0$
  - Exception: friction

- Now multiply $\mathbf{F}_i^{(a)} + \mathbf{f}_i - \dot{\mathbf{p}}_i = 0$ by $\delta\mathbf{r}_i$ and sum over $i$

# D'Alembert's Principle

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i)\delta\mathbf{r}_i = 0$$

"constraint" force is out of the game. You can forget (a)

- Force of constraints dropped out because $\mathbf{f}_i \delta\mathbf{r}_i = 0$
- Called D'Alembert's Principle (1743)

- Now we switch from $\mathbf{r}_i$ to $q_j$

$$\text{1st term} = \sum_i \mathbf{F}_i \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j$$

$$Q_j \equiv \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Generalized force

- Unit of $Q_j$ not always [force]
- $Q_j$ $q_j$ is always [work]

# D'Alembert's Principle

$$\text{2nd term} = \sum_i \dot{\mathbf{p}}_i \delta \mathbf{r}_i = \sum_i \dot{\mathbf{p}}_i \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} m_i \ddot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

- A bit of work can show $\ddot{\mathbf{r}}_i \dfrac{\partial \mathbf{r}_i}{\partial q_j} \rightarrow \dfrac{d}{dt}\left[\dfrac{\partial}{\partial \dot{q}_j}\left(\dfrac{v_i^2}{2}\right)\right] - \dfrac{\partial}{\partial q_j}\left(\dfrac{v_i^2}{2}\right)$

$$= \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

$$T \equiv \sum_i \frac{m v_i^2}{2}$$

- D'Alembert's Principle becomes

$$\sum_j \left\{ \left[ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

# Lagrange's Equations

$$\sum_j \left\{ \left[ \frac{d}{dt}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

These are free

- Generalized coordinates $q_j$ are independent

$$\frac{d}{dt}\left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

Almost there!

- Assume forces are conservative $\mathbf{F}_i = -\nabla_i V$

$$Q_j \equiv \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Throw this back in

# Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0$$

- Assume that $V$ does not depend on $\dot{q}_j$ ➡ $\frac{\partial V}{\partial \dot{q}_j} = 0$

Finally ➡

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$L = T(q_j, \dot{q}_j, t) - V(q_j, t)$$

Done!

# Assumptions We Made

- Constraints are holonomic $\Rightarrow$ $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t)$
  - We always assume this
- Constraint forces do no work $\Rightarrow$ $\mathbf{f}_i \delta \mathbf{r}_i = 0$
  - Forget frictions
- Applied forces are conservative $\Rightarrow$ $\mathbf{F}_i = -\nabla_i V$
  - Lagrange's Eqn. itself is OK if *V* depends explicitly on *t*
- Potential *V* does not depend on $\dot{q}_j$ $\Rightarrow$ $\dfrac{\partial V}{\partial \dot{q}_j} = 0$

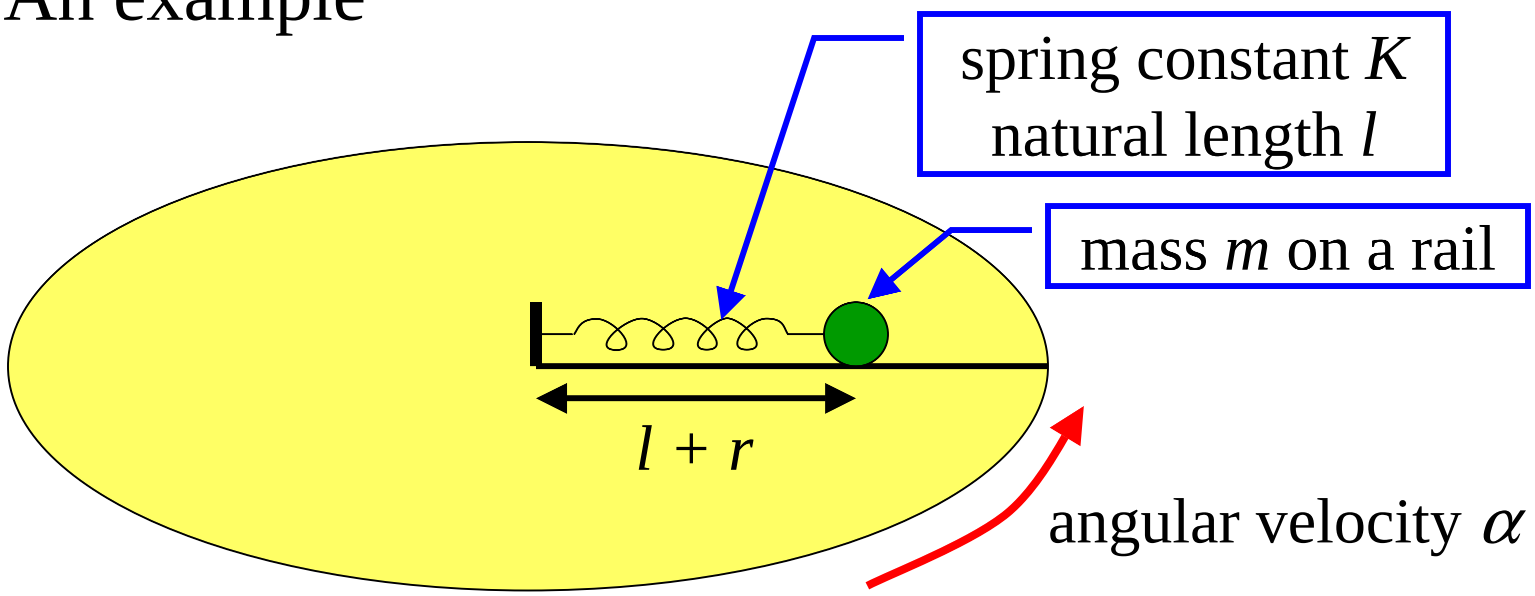
Will review the last assumption later

# Example: Time-Dependent

- Transformation functions may depend on $t$
  - Generalized coordinate system may move
  - E.g. coordinate system fixed to the Earth

$$\mathbf{r}_i = \mathbf{r}_i(q_j, t)$$

- An example



# Example: Time-Dependent

- Transformation functions: $\begin{cases} x = (l + r)\cos \alpha t \\ y = (l + r)\sin \alpha t \end{cases}$
- Kinetic energy $T = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2\right\} = \frac{m}{2}\left\{\dot{r}^2 + (l + r)^2 \alpha^2\right\}$
- Potential energy $V = \frac{K}{2} r^2$

$$\Rightarrow \quad L = \frac{m}{2}\left\{\dot{r}^2 + (l + r)^2 \alpha^2\right\} - \frac{K}{2} r^2$$

Lagrange's Equation $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = m\ddot{r} - m\alpha^2 (l + r) + Kr = 0$

# Example: Time-Dependent

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = m\ddot{r} - m\alpha^2(l + r) + Kr = 0$$

$$m\ddot{r} + (K - m\alpha^2)\left(r - \frac{m\alpha^2 l}{K - m\alpha^2}\right) = 0$$

- If $K > m\alpha^2$, a harmonic oscillator with $\omega = \sqrt{\frac{K - m\alpha^2}{m}}$
  - Center of oscillation is shifted by $\frac{m\alpha^2 l}{K - m\alpha^2}$
- If $K < m\alpha^2$, moves away exponentially
- If $K = m\alpha^2$, velocity is constant
  - Centripetal force balances with the spring force

# Note on Arbitrarity

- Lagrangian is not unique for a given system
  - If a Lagrangian $L$ describes a system

$$L' = L + \frac{dF(q,t)}{dt}$$

works as well for any function F

  - One can prove

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right)\right) - \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) = 0$$

using

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

# Assumptions We Made

- Constraints are holonomic ➡ $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t)$
  - We always assume this
- Constraint forces do no work ➡ $\mathbf{f}_i \delta \mathbf{r}_i = 0$
  - Forget frictions
- Applied forces are conservative ➡ $\mathbf{F}_i = -\nabla_i V$
  - Lagrange's Eqn. itself is OK if *V* depends explicitly on *t*
- Potential *V* does not depend on $\dot{q}_j$ ➡ $\dfrac{\partial V}{\partial \dot{q}_j} = 0$

Let's review the last assumption

# Velocity-Dependent Potential

- We assumed $Q_j = -\frac{\partial V}{\partial q_j}$ and $\frac{\partial V}{\partial \dot{q}_j} = 0$ so that

This had to be 0

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \Rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

  - We could do the same if we had

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad U = U(q_j, \dot{q}_j, t)$$

Generalized, or velocity-dependent "potential"

$$\Rightarrow L = T(q_j, \dot{q}_j, t) - U(q_j, \dot{q}_j, t)$$

# EM Force on Particle

- Lorentz force on a charged particle

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

Velocity-dependent. Can't find a usual potential $V$

  - **E** and **B** fields are given by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

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  - Force is $v$-dependent → Need a $v$-dependent potential

$$U = q\phi - q\mathbf{A} \cdot \mathbf{v}$$ works

check

- Lagrangian is $L = \frac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}$

# Monogenic System

- If all forces in a system are derived from a generalized potential,
  its called a monogenic system

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

  - $U$ is a function of $q, \dot{q}, t$
  - Lorentz force is monogenic
- A monogenic system is conservative only if $U = U(q)$
  - Or $\frac{\partial U}{\partial \dot{q}} = \frac{\partial U}{\partial t} = 0$
- Lagrange's Equation works on a monogenic system

# Hamilton's Principle

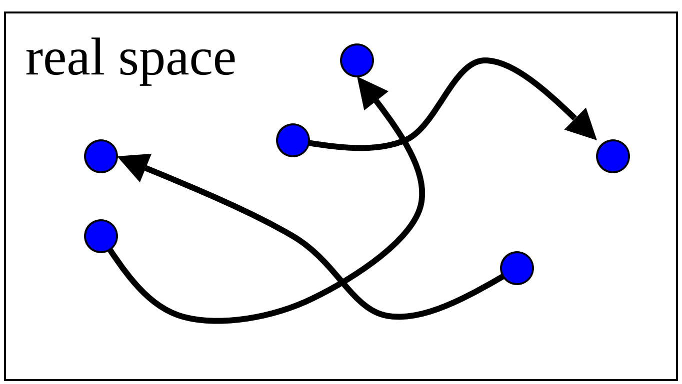
- We derived Lagrange's Eqn from Newton's Eqn using a "differential principle"
  - D'Alembert's principle uses infinitesimal displacements
- It's possible to do it with an "integral principle"

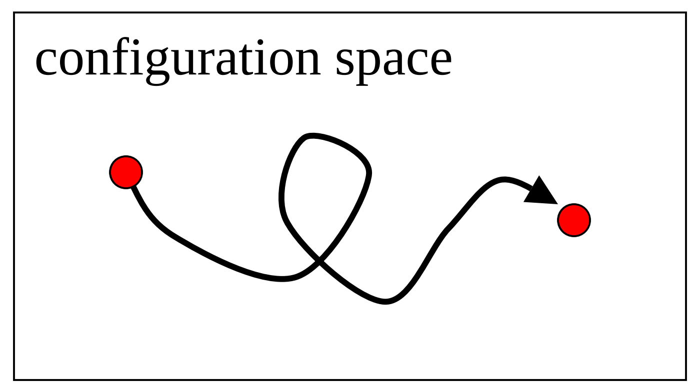
Hamilton's Principle

# Configuration Space

- Generalized coordinates $q_1,\ldots,q_n$ fully describe the system's configuration at any moment
- Imagine an *n*-dimensional space
  - Each point in this space $(q_1,\ldots,q_n)$ corresponds to one configuration of the system
  - Time evolution of the system → A curve in the configuration space

configuration space



# Action Integral

- A system is moving as $q_j = q_j(t) \quad j = 1 \ldots n$
- Lagrangian is $L(q, \dot{q}, t) = L(q(t), \dot{q}(t), t)$

integrate → $I = \int_{t_1}^{t_2} L dt$ Action, or action integral

  - Action $I$ depends on the entire path from $t_1$ to $t_2$
  - Choice of coordinates $q_j$ does not matter
    - Action is invariant under coordinate transformation

# Hamilton's Principle

The action integral of a physical system is *stationary* for the actual path

- This is equivalent to Lagrange's Equations
  - We will prove this

We will also define "stationary"

- Three equivalent formulations
  - Newton's Eqn depends explicitly on *x*-*y*-*z* coordinates
  - Lagrange's Eqn is same for any generalized coordinates
  - Hamilton's Principle refers to no coordinates
    - Everything is in the action integral

Hamilton's Principle is more fundamental

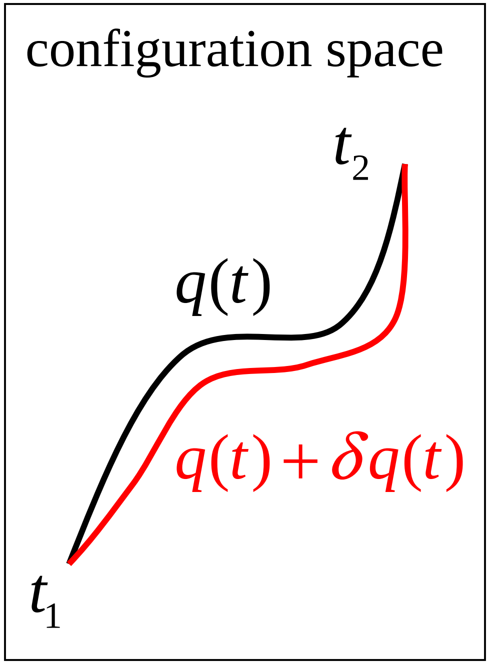
probably...

# Stationary

- Consider two paths that are close to each other
  - Difference is infinitesimal
- Stationary means that the difference of the action integrals is zero to the 1st order of $\delta q(t)$
  - Similar to "first derivative = 0"

$$\delta I = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0$$

- Almost same as saying "minimum"
  - It could as well be maximum



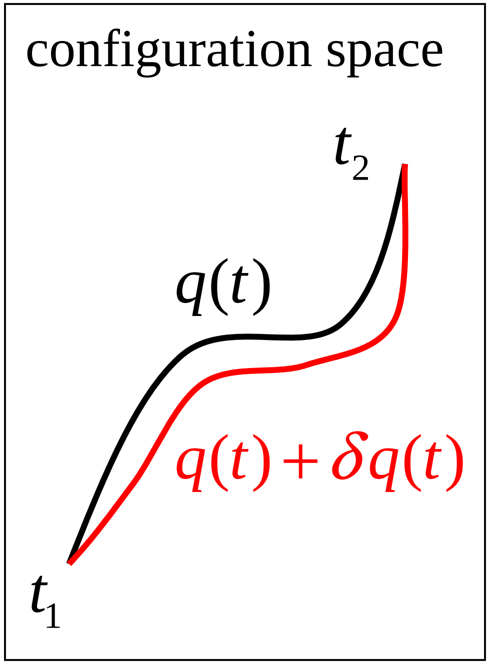

$$\delta q(t_1) = \delta q(t_2) = 0$$

# Infinitesimal Path Difference

- What's $\delta q(t)$?
  - It's arbitrary … sort of
  - It has to be zero at $t_1$ and $t_2$
  - It's well-behaving

Continuous, non-singular, continuous 1st and 2nd derivatives

Don't worry too much




- Have to shrink it to zero
  - Trick: write it as $\delta q(t) = \alpha\eta(t)$
    - $\alpha$ is a parameter, which we'll make → 0
    - $\eta(t)$ is an arbitrary well-behaving function $\eta(t_1) = \eta(t_2) = 0$

# Hamilton → Lagrange

- To derive Lagrange's Eqns from Hamilton's Principle

$$\delta I = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0$$

  - Define $I(\alpha) \equiv \int_{t_1}^{t_2} L(q(t) + \alpha\eta(t), \dot{q}(t) + \alpha\dot{\eta}(t), t)dt$
  - $\delta I$ is then $\lim_{\alpha\to 0}[I(\alpha) - I(0)]$ → $\left(\frac{\partial I}{\partial\alpha}\right)_{\alpha=0} d\alpha$
  - We must show that $\left(\frac{\partial I}{\partial\alpha}\right)_{\alpha=0} = 0$ leads to Lagrange's Eqns

A bit of work. Will do it on Thursday

# Summary

- Derived Lagrange's Eqn from Newton's Eqn
  - Using D'Alembert's Principle ← Differential approach
- Assumptions we made:
  - Constraints are holonomic → Generalized coordinates
  - Forces of constraints do no work → No frictions
  - Other forces are monogenic → Generalized potential
- Introduced Hamilton's Principle
  - Integral approach
  - Defined the action integral and "stationary"
  - Derivation in the next lecture

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$